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Operator Algebras Problem set 3

To be submitted by Wednesday, **April 30**, 4 pm.

Exercise 1

Prove in the following steps that the mapping $t \to \mathrm{ev}_t$, defined by

$$\mathrm{ev}_t : C([0,1]) \to \mathbb{C},\ f \mapsto f(t),$$

defines a homeomorphism Ψ between $[0,1]$ (with the usual topology) and $\mathrm{Spec}(C([0,1])$.

(a) Verify that ev_t is an element of $\mathrm{Spec}(C([0,1]))$ for every $t \in [0,1]$.

(b) Show that Ψ is injective.
Hint: Consider the function $t \mapsto |s-t|$ for any fixed $s \in [0,1]$.

(c) Show that $C([0,1])$ is the unique closed ideal I of $C([0,1])$ with the property that for every $t \in [0,1]$, there exists some $f \in I$ with $f(t) \neq 0$.
Hint: Conclude from the compactness of $[0,1]$ that I contains an invertible element.

(d) Show that the image of $[0,1]$ under Ψ is all of $\mathrm{Spec}(C([0,1]))$.
Hint: Using (c), find for every $\phi \in \mathrm{Spec}(C([0,1]))$ some $t \in [0,1]$ with $\ker(\phi) = \ker(\mathrm{ev}_t)$.

(e) Show that Ψ is continuous.

(f) Show that also the inverse of Ψ is continuous.
Hint: Use the Hausdorff property of $\mathrm{Spec}(C([0,1]))$.

Exercise 2

Let A be a unital C^*-algebra.

(a) Show that for every invertible $x \in A$, we have

$$\mathrm{sp}(x^{-1}) = \{\lambda^{-1};\ \lambda \in \mathrm{sp}(x)\}.$$

(b) Show that the spectrum of a unitary $u \in A$ is contained in the unit circle of $\mathbb{C}$, that is,

$$\mathrm{sp}(u) \subset \{\lambda \in \mathbb{C};\ |\lambda| = 1\}.$$

Talk

The main source for this exercise is: Lecture Notes on C^*-algebras

(a) Read Section 2.1 to 2.3. Give a brief introduction to representation theory of groups, that is, explain what a regular representation is, give the definition a group algebra, and consider the special case of finite groups.

(b) Read Section 2.4. Define the *(full) group C^*-algebra* $C^*(G)$ and the *reduced group C^*-algebra* $C^*_r(G)$ for a discrete group G and state some facts.

(c) Read Section 2.5. What is Pontryagin duality for compact groups? (Consider Theorem 2.5.5) What is $C^*(\mathbb{Z})$?

For c) you may also consult the motivation of Chapter 19 in Functional Analysis 2