Faculty of Mathematics
Moritz Weber
Marcel Scherer
Jonas Metzinger

UNIVERSITÄT DES SAARLANDES
Fakultät für Mathematik und Informatik

Operator Algebras Problem set 2

To be submitted by Wednesday, **April 23**, 4 pm.

Exercise 1 (4 Points)

(a) Prove that Example 2.3 is indeed a C^*-algebra, that is, prove that, for any compact Hausdorff space X, $C(X)$ is a commutative unital C^*-algebra.

(b) Show that for any two compact Hausdorff spaces X and Y and any continuous mapping $h : X \to Y$, the map $\alpha_h : C(Y) \to C(X)$ defined by $f \mapsto f \circ h$ is a $*$-homomorphism.

(c) Prove that in the situation of (b), if h is a homeomorphism, the map α_h is an isometric $*$-isomorphism.

Exercise 2 (4 Points)

Let A be a non-unital C^*-algebra. A *double centralizer* of A is a pair (L, R) of linear maps $L, R : A \to A$ satsifying $L(ab) = L(a)b, R(ab) = aR(b)$ and $aL(b) = R(a)b$ for all $a, b \in A$. The *multiplier algebra* $M(A)$ of A is defined as

$$M(A) = \{\text{(L,R) double centralizer of} A\}$$

with the operations

$$\begin{aligned}(L_1, R_1) + (L_2, R_2) &= (L_1 + L_2, R_1 + R_2)\\ (L_1, R_1) * (L_2, R_2) &= (L_1 L_2, R_2 R_1)\\ (L, R)^* &= (R^*, L^*)\\ \lambda(L, R) &= (\lambda L, \lambda R)\end{aligned}$$

for $\lambda \in \mathbb{C}$ and $(L, R); (L_1, R_1), (L_2, R_2) \in M(A)$. (Note, for a linear map $T : A \to A$, the map $T^* : A \to A$ is defined as $T^*(x)(T(x^*))^*$.) We define a norm on $M(A)$ via

$$\|(L, R)\| = \|L\| + \|R\|$$

for $(L, r) \in M(A)$.

(a) Show that $M(A)$ is a unital C^*-algebra and that the map

$$\phi : A \to M(A), a \mapsto (L_a, R_a),$$

where $L_a(x) = ax$ and $R_a(x) = xa$ for $x \in A$ is an isometric $*$-homomorphism. Furthermore, prove that $\phi(A)$ is an ideal in $M(A)$. Thus every non-unital C^*-algebra can be embedded as an ideal in a unital C^*-algebra.

(b) Prove that $M(A)$ is the largest unitization of A, that is, if B is a unital C^*-algebra and $A \subset B$ as an ideal, then there exists a $*$-homomorphism from B to $M(A)$ which extends the embedding $A \subset M(A)$.

Talk (0 Points)

(a) Explain that $A = \bigoplus_{k=1}^{N} M_{n_k}(\mathbb{C})$ is a C^*-algebra, where $n_k \in \mathbb{N}$ and $M_{n_k}(\mathbb{C})$ is the algebra of $n_k \times n_k$ matrices.

(b) Show (or at least motivate) that any finite-dimensional C^*-algebra is of the form as in (a). This is known as the Theorem of Artin-Wedderburn

Recommended literature: Gerald Murphy "C^*-algebras and Operator Theory", Theorem 6.3.8