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Operator Algebras   Problem set 1

No submission, Discussion on Monday, April 14

Exercise 1 (0 Points)

Let H be a Hilbert space and $U, V, P \in \mathcal{B}(H)$ be operators. Show:

(a) V is an isometry if and only if $\langle Vx, Vy\rangle = \langle x, y\rangle$ for all $x, y \in H$.

(b) U is unitary (i.e. $U^*U = UU^* = id_H$) if and only if U is a Hilbert space isomorphism.

(c) P is a projection if and only if there exists a closed Hilbert space $K \subset H$ such that $P(x + y) = x$ for all $x + y \in K \oplus K^\perp = H$.

Hint: This statement is proposition 1.34 in the ISem24 script. You should either prove the statement yourself or follow the proof given there.

A linear bounded operator V is called *partial isometry* if $VV^*V = V$.

Exercise 2 (0 Points)

Let V be a bounded linear operator on a Hilbert space H. Show that the following statements are equivalent:

(a) V is a partial isometry.

(b) V^*V is a projection.

(c) VV^* is a projection

(d) There exists a Hilbert subspace $K \subset H$ such that $\langle Vx, Vy\rangle = \langle x, y\rangle$ and $Vz = 0$ for all $z \in K^\perp$.

Exercise 3 (0 Points)

Let X be a set. Then $(e_x)_{x\in X}$, defined as $e_x : \ell^2(X) \to \mathbb{C}, (a_y)_{y\in X} \mapsto a_x$ for $x \in X$, is an orthonormal basis of $\ell^2(X)$.

(a) For $X = \mathbb{N}$, there exists a linear bounded operator S on $\ell^2(\mathbb{N})$ with $Se_n = e_{n+1}$ for all $n \in \mathbb{N}$. This operator is called *unilateral shift*. Show:

  (i) The adjoint operator S^* of S sati $S^*e_n = e_{n-1}$ for all $1 \le n \in \mathbb{N}$ and $S^*e_0 = 0$.

  (ii) S is an isometry but not unitary.

(b) For $X = \mathbb{Z}$, the *bilateral shift* $\tilde{S}$ on $\ell^2(\mathbb{Z})$ is defined by $\tilde{S}(e_n) = e_{n+1}$ for $n \in \mathbb{Z}$. Decide, with proof, whether $\tilde{S}$ is unitary.

(c) Define, for $1 \le N \in \mathbb{N}$, an operator on $\ell^2(X)$ for $X = \{1, \dots, N\}$ which can be seen as an analog of $\tilde{S}$.