UNIVERSITÄT DES SAARLANDES

THE K-THEORY OF FREE WREATH PRODUCTS OF COMPACT QUANTUM GROUPS

Bachelor Thesis

by

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Introduction

In this Bachelor thesis, we aim to compute the K-theory of the free wreath product of compact quantum groups with the quantum symmetry group S_N^+, as done 2024 by Fima and Troupel in [FT24]. By this we can also explicitly compute the K-theory of the quantum hyperoctahedral group H_N^+.

In 1936 Murray and von Neumann introduced von Neumann algebras as "rings of operators", [Mur90]. Gelfand and Naimark, in 1943, formalised the concept of C^*-algebras as an abstraction of subalgebras of the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$, [GN43]. Since then the theory of operator algebras, consisting of both of them, has been a fruitful field in functional analysis. As they often come with a non-commutativity, they can be used to describe a physical quantum system. Overall, the theory of C^*-algebras provides us with a "machinery" that allows us to define and understand "non-commutative mathematics", for example such as non-commutative topology and non-commutative geometry, since commutative C^*-algebras are isomorphic to $C(X)$ for some compact set X by the Gelfand-Naimark Theorem.

In the theory of C^*-algebras, K-theory serves as an invariant theory. In 1961 Atiyah and Hirzebruch introduced the topological K-theory [AH61], but it quickly became clear that basic definitions of K-theory are also useful for rings in a more general way. By this, one obtained the algebraic K-theory and the K-theory of C^*-algebras, where the last is more a kind of analogue of the topological K-theory of Atiyah and Hirzebruch. K-theory is useful to determine whether C^*-algebras are not isomorphic to each other.

The theory of "quantum group" will be important in this thesis. The term "quantum group" does not have a single definition but refers to a variety of similar objects, where the common idea underlying these objects is to extend the notion of a group to the realm of non commutative geometry. There are two main approaches to this subject: one is purely algebraic, while the other is more analytical. Our main approach will be mostly analytical via the notion of compact quantum groups, or "compact pseudo groups" which was first introduced in [Wor87] by Woronowicz in 1987 and then was further developed by him in [Wor98]. As Pontryagin developed a duality theory for (locally) compact abelian groups in [Pon34], it turns out that the Pontryagin duality theory on compact quantum groups is a kind of generalisation, since the classical duality fails for non-abelian compact groups. This can somehow be seen as the starting point of Woronowicz. For this one has to look at the representation theory of compact quantum groups.

The first examples of such quantum groups came mostly by "liberation", i.e. dropping the commutativity and by "deformation" of Lie algebras, such as $\mathrm{SU}_q(2)$, where we deform commutativity. Also discrete groups will give us examples, as we will see. However a well-known example for liberation is the quantum symmetry group S_N^+, introduced 1998 by Wang in [Wan98], which can be seen as a generalisation of the space of continuous functions on the symmetric group S_N. Earlier in 1995 Wang also introduced the quantum versions of the orthogonal group O_N and the unitary group U_N in [Wan95].

Another important concept in this thesis is the free wreath product by the quantum symmetric group. Classically, the wreath product of a group G by S_n, denoted $G \wr S_n$, is defined using the natural action of S_N on N copies of G. In analogy to algebraic group theory, Bichon defined the free wreath product of compact quantum groups with the quantum symmetric group S_N^+ in [Bic04]. The free wreath product with amalgamation was then defined by Freslon, [Fre23].

For a long time, it was unclear how operator algebras of this construction of the free wreath product behave, until they were described by Fima and Troupel in [FT24] in 2024. To achieve this, they used graphs of C^*-algebras, which proved to be an effective tool, in full analogy to the Bass-Serre theory for algebraic groups, where graphs of groups were considered. To compute the K-theory of the free wreath product, they also used methods from Kasparov's KK-theory. As a special case, they also determined the K-theory of the free hyperoctahedral quantum group H_N^+.

As an overview of the chapters, the following will serve:

In the first chapter, we want to review the basics regarding C^*-algebras and introduce important knowledge regarding K-theory of C^*-algebras. We will refrain from proofs in this chapter but will refer to sources.

The second chapter will focus on Kasparov's KK-theory as a kind of generalisation of K-theory. A theory that offers many technical hurdles and obstacles but still yields fruitful results. Here, we want to consider KK-theory as a kind of extension of K-theory of C^*-algebras. Many definitions must be formulated particularly at the beginning. The goal of this chapter will be to understand what the KK^0 and KK^1 groups are and what properties they possess. KK-equivalence will also be considered.

In the third chapter, we want to look at compact quantum groups according to Woronowizc [Wor87; Wor98] and their representation theory. A compact quantum group is understood as a unital C^*-algebra A equipped with a comultiplication $\Delta\colon A \to A \otimes A$ fulfilling certain properties. These should be seen as a kind of generalisation of compact groups, even though they are not groups themselves. To understand these and introduce concepts of duality, such as Pontryagin duality for abelian groups, it will show, that it is useful to look at representations. In particular, we will introduce dual discrete compact quantum groups. However, it will also be shown repeatedly that it makes sense to look at some kind of algebraic approach to compact quantum groups, which is done within the theory of Hopf *-algebras. Moreover we introduce the free product and the free wreath product (with amalgamation) of compact quantum groups.

The fourth chapter will be devoted to the theory of graphs of C^*-algebras. The idea will turn out to be a kind of quantum version of classical Bass-Serre theory. We will introduce concepts such as the fundamental C^*-algebra. Essentially, Chapter four will be based on [FF14], which first introduced the theory of graphs of C^*-algebras. Moreover we want to find a way to express the free wreath product as a fundamental C^*-algebra.

In the final chapter, we want to focus on the main results. First, with the help of KK-theory, incorporating the theory of graphs of C^*-algebras, we want to prove the following two 6-term exact sequences, as done in [FG18].

Theorem A. *The following* 6*-term exact sequences hold,*

$$\begin{array}{ccccc}
\bigoplus_{e\in E^+(\mathcal{G})} KK^0(C,B_e) & \xrightarrow{\sum s_e^*-r_e^*} & \bigoplus_{p\in V(\mathcal{G})} KK^0(C,A_p) & \longrightarrow & KK^0(C,P_\bullet) \\
\uparrow & & & & \downarrow \\
KK^1(C,P_\bullet) & \longleftarrow & \bigoplus_{p\in V(\mathcal{G})} KK^1(C,A_p) & \xleftarrow[\sum s_e^*-r_e^*]{} & \bigoplus_{e\in E^+(\mathcal{G})} KK^1(C,B_e),
\end{array}$$

and

$$\begin{array}{ccccc}
\bigoplus_{e\in E^+(\mathcal{G})} KK^0(B_e,C) & \xleftarrow[\sum s_{e*}-r_{e*}]{} & \bigoplus_{p\in V(\mathcal{G})} KK^0(A_p,C) & \longleftarrow & KK^0(P_\bullet,C) \\
\downarrow & & & & \uparrow \\
KK^1(P_\bullet,C) & \longrightarrow & \bigoplus_{p\in V(\mathcal{G})} KK^1(A_p,C) & \xrightarrow{\sum s_{e*}-r_{e*}} & \bigoplus_{e\in E^+(\mathcal{G})} KK^1(B_e,C).
\end{array}$$

Then, we want to use these to compute the K-theory of the reduced and full compact quantum group as done 2024 in [FT24].

Theorem B. *For any compact quantum group G and every integer $N \in \mathbb{N}$ we have,*

$$\begin{aligned}
K_0(C_\bullet(G \wr_* S_N^+)) &\cong K_0(C_\bullet(G)) \otimes \mathbb{Z}^{N^2} \oplus K_0(C_\bullet(S_N^+))/\mathbb{Z}^{N^2} \\
&\cong \begin{cases} K_0(C_\bullet(G))^{\oplus N^2}/\mathbb{Z}^{2N-2} & \text{if } N \neq 3 \\ K_0(C_\bullet(G))^{\oplus N^2}/\mathbb{Z}^3 & \text{if } N = 3 \end{cases}, \\
K_1(C_\bullet(G \wr_* S_N^+)) &\cong K_1(C_\bullet(G))^{\oplus N^2} \oplus K_1(C_\bullet(S_N^+)) \\
&\cong \begin{cases} K_1(C_\bullet(G))^{\oplus N^2} \oplus \mathbb{Z} & \text{if } N \geq 4 \\ K_1(C_\bullet(G))^{\oplus N^2} & \text{if } N \leq 3 \end{cases},
\end{aligned}$$

where $C_\bullet(G)$ denotes either the reduced or full C^*-algebra.

At least, we want to mention some applications and compute some explicit examples.

Contents

Chapter I.

Preliminaries

In this chapter we review basic notions and definitions we need in this bachelor thesis. We will start this chapter with an introduction to C^*-algebras, followed by an small introduction to the K-theory of C^*-algebras.

1. C^*-algebras

In the topic of operator algebras C^*-algebras are one of the main actors beside the von Neumann algebras. This section is mainly based on [Bla06, Ch. 2] and [LVW20], where also all proofs can be found.

Definition 1.1.1 (C^*-algebras): (i) A *Banach algebra* A is a normed $\mathbb{C}$-algebra, which is complete and its norm is submultiplicative, i.e. $\|xy\| \leq \|x\| \|y\|$. A *Banach* **-algebra* A is a Banach algebra with an involution *, i.e. an antilinear map $^*\colon A \to A$ such that $(x^*)^* = x$ and $(xy)^* = y^*x^*$ for all $x, y \in A$.

(ii) A C^**-algebra* is a Banach *-algebra A satisfying the C^*-identity $\|x^*x\| = \|x\|^2$. If A has a unit with respect to the multiplication, we call A *unital*.

(iii) Let A be a C^*-algebra and let $B \subseteq A$ *-subalgebra, i.e. B is closed under addition, (scalar-) multiplication and involution. B is called C^**-subalgebra* if it is norm-closed.

(iv) Let A, B be C^*-algebras, we call a map $\varphi\colon A \to B$ a **-homomorphism*, if φ is linear, multiplicative and if $\varphi(x^*) = \varphi(x)^*$ for all $x \in A$.

We first want to look at some simple examples, before we collect some of the main results for C^*-algebras.

Example 1.1.2: (i) Let X be a compact Hausdorff space, then the space of continuous function

$$C(X) := \{f\colon X \to \mathbb{C} \mid f \text{ is continuous}\}$$

is a unital C^*-algebra with pointwise addition and multiplication and the supremum norm $\|\cdot\|_\infty$. As involution we choose $f^*(x) = \overline{f(x)}$ for all $x \in X$ and for all $f \in C(X)$.

(ii) Any norm closed *-subalgebra of $\mathcal{B}(\mathcal{H})$ is a C^*-algebra, where $\mathcal{B}(\mathcal{H})$ denotes the bounded linear operators on a Hilbert space H. For example the space of compact operators $\mathcal{K}(\mathcal{H})$ is a norm closed *-subalgebra of $\mathcal{B}(\mathcal{H})$.

By analogy with $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, one defines the following in the case of C^*-algebras.

Definition 1.1.3: Let A be a unital C^*-algebra.

(i) An element $p \in A$ is called *projection* if and only if it is self adjoint and idempotent, i.e. $p = p^* = p^2$.

(ii) An element $s \in A$ is called *isometry* if and only if $s^*s = 1$.

(iii) An element $u \in A$ is called *unitary* if and only if $uu^* = u^*u = 1$.

(iv) An element $v \in A$ is called *partial isometry* if and only if $vv^*v = v$.

(v) An element $x \in A$ is called *normal* if and only if $xx^* = x^*x$.

As done in [LVW20; Bla06] one can equip C^*-algebras with some order structure "$\leq$". This order structure is naturally preserved by *-homomorphisms. As a kind of generalisation one defines *(completely) positive maps.*

Definition 1.1.4 ((Completely) Positive maps): Let A and B be C^*-algebras and $\psi\colon A \to B$ be a linear map.

(i) An element $x \in A$ is called *positive* if x is self-adjoint and the spectrum $\mathrm{sp}(x) \subseteq [0, \infty)$, we write $x \geq 0$. Moreover we write $x \geq y$ if $x - y \geq 0$.

(ii) The map ψ is called *positive* if for all $x \in A$ with $x \geq 0$, we have $\psi(x) \geq 0$.

(iii) The map ψ is called *completely positive* if the induced maps

$$\psi_k\colon M_k(A) \to M_k(B), \quad (a_{ij}) \mapsto (\psi(a_{ij}))$$

are positive for all $k \in \mathbb{N}$.

Normally we will work with unital C^*-algebras, but there also exist non-unital C^*-algebras, such as the compact operators on some Hilbert space, but they always possess at least an *approximate unit.*

Definition 1.1.5 (Approximate units): Let A be a C^*-algebra, and $I \subseteq A$ a subset. An *approximate unit* for I is a net $(u_\lambda)_{\lambda\in\Lambda} \subseteq I$ such that

(i) $0 \leq u_\lambda$ and $\|u\|_\lambda \leq 1$ for all $\lambda \in \Lambda$,

(ii) if $\lambda \leq \mu$ then $u_\lambda \leq u_\mu$,

(iii) we have $u_\lambda x \to x$ and $x u_\lambda \to x$ for all $x \in I$.

If A has a countable approximate unit, then A is called *σ-unital*.

There is also a way to embed any C^*-algebra in the maximal unital C^*-algebra containing the C^*-algebra itself as an essential ideal, which will be the so called *multiplier algebra*.

Definition 1.1.6 (Multiplier algebra): Let A be any C^*-algebra. A *double centraliser* is a pair (L, R) of bounded linear maps on A such that $aL(b) = R(a)b$ for all $a, b \in A$. The set of all double centralisers is called *multiplier algebra*, and is denoted by $M(A)$.

A tool we will need are the so called *conditional expectations*, which are in some sense a non-commutative generalisation of conditional expectations in classical probability theory.

Definition 1.1.7 (Conditional Expectations): Let A be a unital C^*-algebra and $B \subseteq A$ a unital C^*-subalgebra. A linear, positive, surjective and unital map $\varphi\colon A \to B$ satisfying $\varphi \circ \varphi = \varphi$ is called *conditional expectation*.

The following theorem states that all commutative C^*-algebras are isomorphic to continuous functions on some compact set.

Proposition 1.1.8 (Gelfand-Naimark Theorem): *Let A be a unital C^*-algebra, then is A commutative if and only if there exists a compact Hausdorff space X with $A \cong C(X)$.*

Out of this Theorem of Gelfand and Naimark we get that the theory of commutative C^*-algebras corresponds to topology, therefore we may view the the theory of noncommutative C^*-algebras as a “noncommutative topology”.

One very powerful tool is the *continuous functional calculus*.

Proposition 1.1.9 (Continuous functional calculus): *Let A be a C^*-algebra and x be a normal element in A. There is an isometric *-isomorphism*

$$\Phi\colon C(\mathrm{sp}(x)) \to C^*(x, 1) \subseteq A$$

mapping $\Phi(\mathrm{id}) = x$ and $\Phi(1) = 1$, where $\mathrm{sp}(x)$ denotes the spectrum of x, and $C^*(x, 1)$ denotes the norm-completion of the set of all noncommutative polynomials in x and x^*.

For non-commutative C^*-algebras we also obtain an analogue of the Gelfand-Naimark Theorem. To do this we have to look at the GNS-construction.

Definition 1.1.10 (State and Representation): Let A be a C^*-algebra.

(i) A positive linear functional $\varphi\colon A \to \mathbb{C}$ with $\|\varphi\| = 1$ is called *state*.

(ii) Let $\mathcal{H}$ be a Hilbert space. A *-homomorphism $\pi\colon A \to \mathcal{B}(\mathcal{H})$ is called a *representation* of A on $\mathcal{H}$. It is called *cyclic* if and only if there exists $x \in A$ such that $\overline{\pi(A)x} = H$. We say that x is a *cyclic vector*.

Proposition 1.1.11 (Gelfand-Naimark-Segal): *Let A be a C^*-algebra and $\varphi\colon A \to \mathbb{C}$ be a state. Then there exist a Hilbert space $\mathcal{H}_\varphi$, a representation $\pi_\varphi\colon A \to \mathcal{B}(\mathcal{H}_\varphi)$ and a cyclic vector ξ_φ, such that $\varphi(a) = \langle \pi_\varphi(a)\xi_\varphi, \xi_\varphi\rangle$ for all $a \in A$.*

We often call the triple $(\mathcal{H}_\varphi, \pi_\varphi, \xi_\varphi)$ the *GNS-construction* of φ.

As a consequence we get the following main theorem for noncommutative C^*-algebras.

Corollary 1.1.12 (Second Gelfand-Naimark Theorem): *Let A be a C^*-algebra, then it possesses a faithful representation $\pi\colon A \to B(H)$, i.e. injective representation, for some Hilbert space H. Thus A is isomorphic to some C^*-subalgebra of $B(H)$.*

In the following we want to construct *universal C^*-algebras* which are prescribed by a *set of generators* E and a *relations* $R \subseteq P(E)$, where $P(E)$ is the *set of noncommutative polynomials* in elements in E. For more details have a look at [LVW20, Chapter 6].

Construction 1.1.13: Let I be some index set and $E = \{x_i \mid i \in I\}$ be a set of *generators*. Define $P(E)$ as the set of noncommutative polynomials in elements of E and let $R \subseteq P(E)$ be a set of *relations*. Denote by $I(R)$ the two-sided ideal generated by the relations.

The quotient $A(E|R) = P(E)/I(R)$ is the *universal *-algebra generated by E and R.*

This construction leads us to the following definition, where we construct the *universal C^*-algebra* by choosing a suitable norm, as we will define.

Definition 1.1.14: Let everything be such as in Construction 1.1.13.

(i) We call a map p a *C^*-seminorm* if and only if it is a seminorm and fulfils $p(x^*x) = p(x)^2$. Define

$$\|x\| := \sup\{p(x) \mid p \text{ is a } C^*\text{-seminorm on } A(E|R)\}.$$

(ii) If $\|x\| < \infty$ for all $x \in A(E|R)$ then the *universal C^*-algebra by E and R* defined by

$$C^*(E|R) := \overline{A(E|R)/\{x \in A(E|R) \mid \|x\| = 0\}}^{\|\cdot\|}$$

exists.

It is necessary, when having such a set of generators E and relations R, to check whether the universal C^*-algebra exists and if it is not trivial. To show that it is not trivial, you often use the following *universal property* for universal C^*-algebras.

Proposition 1.1.15 (Universal property): *Let $E = \{x_i \mid i \in I\}$ be a set of generators and $R \subseteq P(E)$ a set of relations, such that the universal C^*-algebra exists. Let B be a C^*-algebra containing a subset $E' = \{y_i \mid i \in I\}$. If the elements in E' satisfy the relations in R, then there exists a unique *-homomorphism $\varphi\colon C^*(E|R) \to B$ mapping x_i to y_i for all $i \in I$.*

2. K-theory of C^*-algebras

In this section we want to briefly introduce the basic definitions and properties of K-theory of C^*-algebras. This section is mainly based on [Bla06].

The K-theory of C^*-algebras is used to distinguish two given C^*-algebras. As one will see we will have two K-groups, K_0 "counts" in some sense the projections, while K_1 "counts" in some sense the unitaries.

Definition 1.2.1: Let A be a C^*-algebra. Two projections $p, q \in A$ are called Murray-von Neumann-equivalent if there exists $v \in A$ such that $v^*v = p$ and $vv^* = q$, write $p \sim q$. This defines an equivalence relation. We then set

$$H(A) = \{[p] \mid p \in M_\infty(A) \text{ is a projection }\},$$

where we denote $M_\infty(A) = \bigcup_{n \in \mathbb{N}} M_n(A)$ with canonical embedding

$$M_n(A) \to M_{n+1}(A), x \mapsto \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix},$$

and where $[p]$ is the equivalence class with respect to Murray-von Neumann-equivalence.

Remark 1.2.2: (i) Indeed one can check, that we also can use instead of Murray-von Neumann equivalence the notion of unitary equivalence or homotopy.

(ii) Moreover $H(A)$ is an abelian semigroup via $[p] + [q] = [p' + q']$, where $p \sim p'$, $q \sim q'$ and $p'q' = 0$.

As mentioned $H(A)$ defines for a C^*-Algebra a semigroup, we now want to get a group by using the so called *Grothendieck construction.*

Definition 1.2.3 (The K_0 group): Let A be a unital C^*-algebra, and consider the diagonal $\Delta = \{([p], [p]) \mid [p] \in H(A)\}$. Then define

$$K_0(A) := G(H(A)) := H(A) \times H(A)/\Delta$$

as the K_0-group of A. Write $[p] - [q]$ for an element in $K_0(A)$.

We now want to collect some basic properties about the functor K_0, which are useful to compute K_0 for concrete examples.

Proposition 1.2.4: *Let A be a unital C^*-algebra. Then the following properties for K_0 hold*

(i) *The map*

$$K_0\colon \{\text{unital } C^*\text{-algebras}\} \to \{\text{abelian groups}\}$$

is a covariant functor.

(ii) *The functor K_0 is additive, i.e. $K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$.*

(iii) *The functor K_0 is finitely stable, i.e. $K_0(M_n(A)) \cong K_0(A)$.*

(iv) *The functor K_0 is homotopy invariant, that means if α is homotopic to β, then $K_0(\alpha) = K_0(\beta)$ and if A is homotopic to B, then $K_0(A) \cong K_0(B)$, where $\alpha, \beta\colon A \to B$ are unital *-homomorphisms.*

In the case of non-unital C^*-algebras, one also can construct a K-theory, which will extend the definition of the unital case.

Remark 1.2.5: Let A be a not necessarily unital C^*-algebra. Then we can define $K_0(A)$ by setting $K_0(A) := \ker K_0(\sigma) \subseteq K_0(\tilde{A})$, where $\tilde{A}$ denotes the minimal unitalisation of A and

$$\sigma\colon \tilde{A} \to \mathbb{C}, (x, \lambda) \mapsto \lambda.$$

One can easily verify, that for a unital C^*-algebra A the definitions of $K_0(A)$ and $K_0(A)$ coincide.

Instead of defining K_1 by equivalence relations on the unitaries (see for instance [Bla98, Ch. 8]), we define K_1 via the suspension functor.

Definition 1.2.6 (Suspension functor): (i) Let A be a C^*-algebra, then define the suspension of A as $SA := C_0((0,1), A)$, where $C_0((0,1), A)$ is the C^*-algebra of continuous functions $f\colon (0,1) \to A$ such that $f(0) = f(1) = 0$.

(ii) Let A, B be C^*-algebras and $\phi\colon A \to B$ be a *-homomorphism, then define

$$S\phi\colon SA \to SB,\ (S\phi)(f)(t) = \phi(f(t)).$$

It is obviously a *-homomorphism. It is clear that S defines a covariant functor.

We now can define the K_1-group.

Definition 1.2.7 (The K_1 group): For a C^*-algebra A, we define by $K_1(A) = K_0(SA)$ the K_1- *group* of A.

It is also possible to define $K_n(A) := K_0(S^n A)$.

Remark 1.2.8: (i) All properties for K_0 from Proposition 1.2.4 also hold for K_1.

(ii) By *Bott periodicity* ([Bla98, Ch. 9.4]) there exist a natural isomorphism $K_{n+2}(A) \cong K_n(A)$, therefore our K-theory for C^*-algebras is fully described by $K_0(A)$ and $K_1(A)$.

By using Bott periodicity we also obtain the following result, the so called *six-term cyclic sequence*.

Proposition 1.2.9: *Let $0 \to I \to A \to A/I \to 0$ be a exact sequence of C^*-algebras, then the following six-term cyclic sequence is exact*

$$\begin{array}{ccccc} K_0(I) & \xrightarrow{K_0\iota} & K_0(A) & \xrightarrow{K_0\pi} & K_0(A/I) \\ \uparrow & & & & \downarrow \\ K_1(A/I) & \xleftarrow[K_1\pi]{} & K_1(A) & \xleftarrow[K_1\iota]{} & K_1(I). \end{array}$$

The maps $K_0(A/I) \to K_1(I)$ and $K_1(A/I) \to K_0(I)$ are constructed by using the Bott maps, see [Bla98, Ch. 9.4].

It turns out that using 6-term exact sequences in general is a useful tool to compute K-theory, as we will also see later in this thesis, as we will construct one for KK-theory in the setting of graph of C^*-algebras.

To conclude we want to give some simple examples for the K-theory of some C^*-algebras.

Example 1.2.10: (i) Let $A = \mathbb{C}^N$ for some $N \in \mathbb{N}$, then $K_0(A) \cong \mathbb{Z}^N$. This follows directly from the fact that K_0 is additive and since $H(\mathbb{C}) = \mathbb{N}_0$. Indeed two projections in $M_N(\mathbb{C})$ are Murray-von Neumann equivalent if and only if the rank is equal.

Moreover deduce $K_1(A) = \{0\}$ again by the additivity of K_1 and the fact $K_1(\mathbb{C}) \cong K_0(S\mathbb{C}) = K_0(C_0((0,1), \mathbb{C}))$. The only continuous projection $C_0((0,1), \mathbb{C})$ is the trivial one.

(ii) From (i) we can also conclude for all $N \in \mathbb{N}$ that $K_0(\bigoplus_{i=1}^N M_{n_i}(\mathbb{C})) = \mathbb{Z}^N$ and $K_1(\bigoplus_{i=1}^N M_{n_i}(\mathbb{C})) = \{0\}$.

(iii) One can easily construct a homotopy $A \sim_h C([0,1], A)$. By the homotopy invariance of K_0 and K_1 we obtain that $K_0(C([0,1], A)) = K_0(A)$ and $K_1(C([0,1], A)) = K_1(A)$. Especially $K_0(C([0,1])) = \mathbb{Z}$ and $K_1(C([0,1])) = \{0\}$.

Chapter II.

KK-theory of C^*-algebras

In this chapter we want to develop basics of *Kasparov's KK-theory*, which will be mainly seen as a generalisation of the classical K-theory. The theory is more technical, which is why we must first introduce basic concepts..The main idea, as in K-theory, will be that we choose a suitable set and then define equivalence relations on it.

We will mainly follow [Bla98] and [JT91], which both are based on the original article of Kasparov from 1981 [Kas80].

1. Kasparov modules

First we want to define the notion of $\mathbb{Z}/2\mathbb{Z}$-graded C^*-algebras and Hilbert-C^*-modules to then define Kasparov modules. This section is only used to introduce the necessary concepts and definitions. Therefore there will be no proofs, which can be mostly found in [Bla98; JT91].

For this chapter let A and B be C^*-algebras.

Definition 2.1.1 (($\mathbb{Z}/2\mathbb{Z}$-)Graded C^*-algebras): Let A be a C^*-algebra.

(i) A *($\mathbb{Z}/2\mathbb{Z}$-)grading* on A is a decomposition $A = A^{(0)} \oplus A^{(1)}$, where $A^{(0)}$ and $A^{(1)}$ are self-adjoint closed linear subspaces of A, such that for all $x \in A^{(n)}, y \in A^{(m)}$ we have $xy \in A^{(n+m)}$, where $n+m$ is the addition of n and m in $\mathbb{Z}/2\mathbb{Z}$.

(ii) Let A be a graded C^*-algebra. The degree ∂x of an element $x \in A^{(n)}$ is defined as $\partial x = n$. Moreover we call elements $x \in A^{(0)} \cup A^{(1)}$ *homogeneous*.

(iii) If there is a self-adjoint unitary $g \in M(A)$ in the multiplier algebra of a graded C^*-algebra such that $A^{(n)} = \{a \in A \mid gag^* = (-1)^n a\}$ for $n = 0, 1$, then the grading is called *even*.

(iv) A C^*-subalgebra B of a graded C^*-algebra A is a *graded C^*-subalgebra*, if $B = (B \cap A^{(0)}) + (B \cap A^{(1)})$.

(v) Let A, B be graded C^*-algebras. A *-homomorphism $\varphi\colon A \to B$ is called graded, if $\varphi(A^{(n)}) \subseteq B^{(n)}$ for $n = 0, 1$.

(vi) We define the *graded commutator* for $a \in A^{(i)}$ and $b \in A^{(j)}$ by

$$[a, b] = ab - (-1)^{ij} ba.$$

Remark 2.1.2 (Alternative definition via $\mathbb{Z}/2\mathbb{Z}$-action): We can also define graded C^*-algebras via a $\mathbb{Z}/2\mathbb{Z}$-action on A:

A graded C^*-algebra A is a C^*-algebra with an order two *-automorphism β_A. The *-automorphism is called *grading automorphism*. When A is graded by $\beta_A\colon A \to A$, then A decomposes in eigenspaces of β_A, with $A^{(0)} = \{a \in A \mid \beta_A(a) = a\}$ and $A^{(1)} = \{a \in A \mid \beta_A(a) = -a\}$. One can see that this is indeed equivalently to our definition of graded C^*-algebras. In case of $A^{(1)} = \{0\}$, we say the grading is *trivial*.

A graded *-homomorphism $\varphi\colon A \to B$ for two graded C^*-algebras A, B with grading automorphisms β_A, β_B satisfies $\varphi \circ \beta_A = \beta_B \circ \varphi$.

Example 2.1.3: (i) Let A be any C^*-algebra. Then $M_2(A)$ has a canonical grading, where all elements of $M_2(A)^{(0)}$ are diagonal matrices, and $M_2(A)^{(1)}$ are the matrices with zero diagonal.

(ii) Let B be a C^*-algebra, we can define an order two *-automorphism on $B \oplus B$ by $\beta_{B\oplus B}(x, y) = (y, x)$. This grading is called *odd grading*, and we denote the odd graded C^*-algebra by $B_{(1)}$.

Now we want to define a tensor product of graded C^*-algebras, that respects the grading in some sense.

Construction 2.1.4 (Maximal tensor product of graded C^*-algebras): Let A and B be graded C^*-algebras, write $A \odot B$ for their algebraic tensor product. For homogeneous elementary tensors we may define a new product and involution on $A \odot B$ by

$$(a_1 \hat{\otimes} b_1)(a_2 \hat{\otimes} b_2) = (-1)^{\partial b_1 \partial a_2} a_1 a_2 \hat{\otimes} b_1 b_2,$$
$$(a \hat{\otimes} b)^* = (-1)^{\partial a \partial b} a^* \hat{\otimes} b^*.$$

We denote $A \hat{\odot} B$ for the *-algebra, we get by this product and involution.

Define $A \hat{\otimes}_{\max} B$ as the *universal enveloping* C^*-algebra of $A \hat{\odot} B$, i.e. separation and norm-completion. We call $A \hat{\otimes}_{\max} B$ the *maximal graded tensor product*.

Construction 2.1.5: Let be everything such as in 2.1.4. Let ϕ and ψ be states, vanishing on $A^{(1)}$ respectively on $B^{(1)}$, of A and B, then $\phi \hat{\otimes} \psi$ is a state on $A \hat{\odot} B$. Then the GNS-representation from $\phi \hat{\otimes} \psi$ gives us a C^*-seminorm on $A \hat{\odot} B$.

As in the case of (ungraded) C^*-algebras the supremum of all this C^*-seminorms is a norm. Taking the completion with respect to this norm, yields us the *minimal graded tensor product*, denoted by $A \hat{\otimes}_{\min} B$ or simply $A \hat{\otimes} B$.

Remark 2.1.6: We get a canonical grading on $A\hat{\otimes}B$ by using 2.1.2. If β_A is a $\mathbb{Z}/2\mathbb{Z}$-action on A and β_B is a $\mathbb{Z}/2\mathbb{Z}$-action on B, then we observe $\beta_A\hat{\otimes}\beta_B$ is a $\mathbb{Z}/2\mathbb{Z}$-action on $A\hat{\otimes}B$. We also denote it by $A\hat{\otimes}B$.

To define Kasparov modules, we need to define Hilbert C^*-modules which generalise classical Hilbert spaces, since we consider a C^*-valued inner product on it.

Definition 2.1.7 (Hilbert C^*-module): Let B be a C^*-algebra. A *pre-Hilbert B-module* is a right B-module E (with complex vector space structure) with a B-valued inner product $\langle\cdot,\cdot\rangle\colon E\times E\to B$ such that

(i) $\langle\cdot,\cdot\rangle$ is sesquilinear,

(ii) $\langle x, yb\rangle = \langle x,y\rangle b$ for all $x,y\in E$ and $b\in B$,

(iii) $\langle y,x\rangle = \langle x,y\rangle^*$ for all $x,y\in E$,

(iv) $\langle x,x\rangle \geq 0$ for all $x\in E$ and if $\langle x,x\rangle = 0$, then $x=0$.

For $x\in E$, set $\|x\| = \|\langle x,x\rangle\|^{1/2}$, which defines a norm on E. If E is complete with respect to this norm, then E is called *Hilbert B-module.*

Moreover we define $\langle E,E\rangle = \overline{\{\langle x,y\rangle \mid x,y\in E\}}^{\|\cdot\|}$ as the *support* of E. If $\langle E,E\rangle = B$, then E is called *full.*

It should be clear out of context, which norm on which space we are actually taking.

We also have an analogue of the Cauchy-Schwarz inequality in the case of Hilbert C^*-modules.

Lemma 2.1.8: *Let E be a pre-Hilbert B-module, and set $\|e\| = \|\langle e,e\rangle\|^{\frac{1}{2}}$for $e\in E$. Then E is a normed vector space, and the following inequalities hold:*

$$\|eb\| \leq \|e\|\,\|b\|\,, \quad e\in E,\, b\in B,$$
$$\|\langle e,f\rangle\| \leq \|e\|\,\|f\|\,, \quad e,f\in E.$$

Example 2.1.9: (i) Let $(E_i)_{i\in I}$ be a family of pre-Hilbert B-modules, then the direct sum $\bigoplus E_i$ is a pre-Hilbert B-module with

$$\langle \oplus x_i, \oplus y_i\rangle = \sum \langle x_i, y_i\rangle.$$

If I is finite and all E_i are Hilbert B-modules, then also $\bigoplus E_i$ is a Hilbert B-module.

(ii) As a special case, taking $E_i = B$ in (i), denote by $\mathbb{H}_B$ the completion of the direct sum of countably many copies of B, i.e. for all $(b_n)\subseteq \mathbb{H}_B$ the series $\sum_n b_n^* b_n$ converges, where the inner product is given by

$$\langle (a_n),(b_n)\rangle = \sum_n \langle a_n, b_n\rangle.$$

We call $\mathbb{H}_B$ the *Hilbert space over B.*

It turns out that the space of bounded operators of Hilbert C^*-modules is too large, therefore we need a substitute, which we will now define.

Definition 2.1.10: Let E_1, E_2 be Hilbert B-modules. Denote by $\mathcal{B}_B(E_1, E_2)$ the set of all B-module homomorphisms $T\colon E_1 \to E_2$, such that there is an adjoint B-module homomorphism $T^*\colon E_2 \to E_1$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E_1$ and $y \in E_2$.
Set $\mathcal{B}_B(E) := \mathcal{B}_B(E, E)$ for a Hilbert B-module E.

We also want a suitable subspace of the compact operators.

Remark 2.1.11: Note that operators T in $\mathcal{B}_B(E)$ are bounded by the closed graph theorem. Indeed since the existence of an adjoint T^* implies that the graph of T must be closed. Thus $\mathcal{B}_B(E) \subseteq \mathcal{B}(E)$. Moreover $\mathcal{B}_B(E)$ is closed with respect to the operator norm in $\mathcal{B}(E)$.
Additional $\mathcal{B}_B(E)$ is a C^*-algebra. It is clear that $\mathcal{B}_B(E)$ is a *-algebra and that $\|ST\| \leq \|S\|\,\|T\|$. The C^*-identity is fulfilled since for all $x \in E$ with $\|x\| \leq 1$ we have by Cauchy-Schwarz

$$\|Tx\|^2 = \|\langle Tx, Tx\rangle\| = \|\langle x, T^*Tx\rangle\| \leq \|T^*Tx\| \leq \|T^*T\|.$$

Thus $\|T\|^2 \leq \|T^*T\|$, and the other inequality is clear since * is isometric.

Definition 2.1.12: Let E_1 and E_2 be Hilbert B-modules. Let $\theta_{x,y}\colon E_1 \to E_2$ for $x \in E_2, y \in E_1$ defined by $\theta_{x,y}(z) = x\langle y, z\rangle$ for $z \in E_2$. We say $\theta_{x,y}$ is a *finite-rank operator*, moreover it has rank 1.
The closure of the span of these operators in $\mathcal{B}_B(E_1, E_2)$ is denoted by $\mathcal{K}_B(E_1, E_2)$. As always we denote $\mathcal{K}_B(E) := \mathcal{K}_B(E, E)$ for a Hilbert B-module E.

In spirit of Remark 2.1.2 we now want to define *graded Hilbert C^*-modules* via *grading automorphisms*.

Definition 2.1.13 (Graded Hilbert C^*-module): Let B be a graded C^*-algebra. A *graded Hilbert B-module E* is a Hilbert B-module equipped with a linear bijection $S_E\colon E \to E$, satisfying

- $S_E(\xi b) = S_E(\xi)\beta_B(b)$ for all $\xi \in E, b \in B$,
- $\langle S_E(\xi_1), S_E(\xi_2)\rangle = \beta_B(\langle \xi_1, \xi_2\rangle)$ for all $\xi_1, \xi_2 \in E$,
- $S_E^2 = \mathrm{id}$.

We then obtain $E^{(0)} := \{\xi \in E \mid S_E(\xi) = \xi\}$ and $E^{(1)} := \{\xi \in E \mid S_E(\xi) = -\xi\}$ such that $E = E^{(0)} \oplus E^{(1)}$.

Example 2.1.14: Let B be a C^*-algebra.

(i) As in Example 2.1.3, a Hilbert B-module E can be trivially graded by taking $S_E = \mathrm{id}$.

(ii) Let B be a graded C^*-algebra, then B as Hilbert B-module is also a graded Hilbert B-module via $S_B = \beta_B$.

(iii) Let E and F be graded Hilbert B-modules, then the direct sum $E \oplus F$ can be graded via $S_E \oplus S_F$ defined by $S_E \oplus S_F(e, f) = (S_E(e), S_F(f))$ for all $e \in E, f \in F$.

Later it will be useful to have a tensor product of Hilbert C^*-modules.

Construction 2.1.15 (Tensor product of Hilbert C^*-modules): Let E and F be Hilbert C^*-modules with respect to B respectively A, and $\phi\colon A \to \mathcal{B}_B(F)$ be a *-homomorphism. Regard F as a left B-module via ϕ, then denote by $E \odot_\phi F$ the algebraic tensor product. This is a right A-module, and one can define the A-valued pre-inner product via $\langle x_1 \otimes x_2, y_1 \otimes y_2 \rangle := \langle x_2, \phi(\langle x_1, y_1 \rangle) y_2 \rangle$.

The universal enveloping C^*-algebra of $E \odot_\phi F$ with respect to $\langle \cdot, \cdot \rangle$ is called the *(internal) tensor product of Hilbert C^*-modules.*

Note that there is a *-homomorphism $j\colon \mathcal{B}_B(E) \to \mathcal{B}_B(E \otimes_\phi F)$ given by

$$j(m)(x \otimes_\phi y) = m(x) \otimes_\phi y$$

for $x \in E$, $y \in F$ and $m \in \mathcal{B}_B(E)$.

The following lemma is proven in [JT91].

Lemma 2.1.16: *In the setting of Construction 2.1.15 the map j maps $\mathcal{K}_B(E)$ to $\mathcal{K}_B(E \otimes_\phi F)$. Let $f\colon B \to A$ be a *-homomorphism, then $m \otimes_\phi \mathrm{id} \in \mathcal{K}_A(E \otimes_f A)$, whenever $m \in \mathcal{K}_B(E)$.*

We now can define *Kasparov modules*, and by this also the set we will later equip with some suitable equivalence relations, to obtain the KK-groups.

Definition 2.1.17 (Kasparov modules): Let A and B be graded C^*-algebras. Define $\mathbb{E}(A, B)$ as the set of all triples (E, ϕ, F), where E is a countably generated (as B-module) graded Hilbert B-module, ϕ is a graded *-homomorphism from A to $\mathcal{B}_B(E)$ and F is an operator in $\mathcal{B}_B(E)$ of rank 1, such that

- $[F, \phi(a)]$,
- $(F^2 - 1)\phi(a)$,
- $(F - F^*)\phi(a)$

are in $\mathcal{K}_B(E)$ for all $a \in A$. The elements of $\mathbb{E}(A, B)$ are called *Kasparov A, B-modules.*

Moreover denote by $\mathbb{D}(A, B) \subseteq \mathbb{E}(A, B)$ the set of triples (E, ϕ, F) such that $[F, \phi(a)] = (F^2 - 1)\phi(a) = (F - F^*)\phi(a) = 0$ for all $a \in A$. An element in $\mathbb{D}(A, B)$ is called *degenerate Kasparov module.*

In the next lemma we want to prove, that we can equip $\mathbb{E}(A, B)$ with the direct sum.

Lemma 2.1.18: *Let $\mathcal{E}_i = (E_i, \phi_i, F_i) \in \mathbb{E}(A, B)$ be Kasparov A, B-modules for $i = 1, \ldots, n$. Then*

$$\bigoplus_i \mathcal{E}_i := (\oplus_i E_i, \oplus_i \phi_i, \oplus_i F_i)$$

is also a Kasparov A, B-module.

Proof. Note that $E := \oplus_i E_i$ is a Hilbert B-module, by Example 2.1.9, and obviously E is countably generated. For the grading we set S_E as in Example 2.1.14, which is

$$S_E(e_1, \ldots, e_n) = (S_{E_1}(e_1), \ldots, S_{E_n}(e_n)),$$

for $e_i \in E_i$. Thus E is a graded Hilbert B-module.

Define $\phi := \oplus_i \phi_i \colon A \to \mathcal{B}_B(E)$ via

$$\phi(a) = \oplus_i \phi_i(a),$$

for $a \in A$ and define $F := \oplus_i F_i \colon A \in \mathcal{B}_B(E)$ by setting

$$F(e_1, \ldots, e_n) = (F_1 e_1, \ldots, F_n e_n),$$

for $e_i \in E_i$. Since all $F_i \in \mathcal{B}_B(E_i)$ are of rank 1, we obtain that $F \in \mathcal{B}_B(E)$ is also of rank 1.

By construction all other properties are fulfilled, as one easily can check. □

2. Definition of the KK-groups and the Kasparov product

Similar as for classical K-theory, we now equip the set of Kasparov modules with equivalence relations. Here we will have a look at homotopy and operator homotopy, but we will not go into further details regarding operator homotopy.

Let A, B, C, D be graded C^*-algebras in this section.

Firstly we have to construct *pushouts* of Hilbert C^*-modules and of Kasparov A, B-modules.

Construction 2.2.1 (Pushout of Hilbert C^*-modules): Let E be a Hilbert A-module and $f \colon A \to B$ be a surjective *-homomorphism. We define a Hilbert submodule

$$N_f := \{e \in E \mid f(\langle e, e \rangle) = 0\},$$

and set $E'_f = E/N_f$ with quotient map $\pi \colon E \to E'_f$.

Define $\pi(e)f(a) = \pi(ea)$ for $e \in E$ and $a \in A$ and as A-valued inner product on E'_f define $\langle \pi(e_1), \pi(e_2) \rangle := f(\langle e_1, e_2 \rangle)$. This defines us a pre-Hilbert A-module. Denote E_f for the completion with respect to the induced norm.

The Hilbert A-module E_f is called *pushout*.

Now we want to do the same on Kasparov A, B-modules.

Construction 2.2.2 (Pushout of Kasparov modules): Let $\mathcal{E} = (E, \phi, F) \in \mathbb{E}(A, B)$ and $\psi\colon B \to C$ be a surjective graded *-homomorphism. Set E_ψ as defined in Construction 2.2.1. We can define a grading automorphism on E'_ψ via

$$S_{E'_\psi}(\pi(x)) = \pi(S_E(x))$$

for $x \in E$ and can extend this to a grading automorphism on E_ψ. Similarly for $F \in \mathcal{B}_B(E)$ we can define $F_\psi \in \mathcal{B}_C(E_\psi)$ by defining

$$F_\psi(\pi(x)) = \pi(F(x))$$

for $x \in E$ on E'_ψ and then extending it by continuity to E_ψ. Moreover since $F \mapsto F_\psi$ is a *-homomorphism one can easily check for $T = \theta_{x,y},\ x, y \in E$ that F_ψ is also a rank 1 operator.

Lastly define $\phi_\psi\colon A \to \mathcal{B}_C(E_\psi)$ via $a \mapsto \phi(a)_\psi$ similarly as we did it for F and F_ψ.

By this we obtain a Kasparov A, C-module $\mathcal{E}_\psi := (E_\psi, \phi_\psi, F_\psi)$, the so called *pushout*.

We also can define the pullback of Kasparov modules.

Construction 2.2.3 (Pullback): Let $\mathcal{E} = (E, \phi, F) \in \mathbb{E}(A, B)$ and $\psi\colon C \to A$ be a graded *-homomorphism. Then $(E, \phi \circ \psi, F)$ is a Kasparov C, B-module, the so called *pullback* $\psi^*(\mathcal{E})$.

Definition 2.2.4 (Isomorphism of Kasparov modules): Let $\mathcal{E}_i = (E_i, \phi_i, F_i)$ be Kasparov A, B-modules for $i = 1, 2$. We say $\mathcal{E}_1$ and $\mathcal{E}_2$ are *isomorphic* if there exist an isomorphism of Hilbert B-modules $\varphi\colon E_1 \to E_2$ such that $S_{E_2} \circ \varphi = \varphi \circ S_{E_1}$, $F_2 \circ \varphi = \varphi \circ F_1$ and $\phi_2(a) \circ \varphi = \varphi \circ \phi_1(a)$ for $a \in A$.

Write $\mathcal{E}_1 \cong \mathcal{E}_2$ in this case.

Set $IB := C([0,1], B) \cong B \otimes C([0,1])$. We can grade IB by taking $\beta_B \otimes \mathrm{id}$ as the grading automorphism, and let π_t be the surjective *-homomorphism $IB \to B$ obtained by evaluation at t. The maps $(\pi_t)_t$ are also graded *-homomorphisms.

Moreover note that if $\mathcal{E}$ is a Kasparov A, IB-module, then its pushout $\mathcal{E}_{\pi_t}$ is a Kasparov A, B-module for all $t \in [0,1]$.

Definition 2.2.5 (Homotopy): Let $\mathcal{E}, \mathcal{F} \in \mathbb{E}(A, B)$. We say that $\mathcal{E}$ and $\mathcal{F}$ are *homotopic* if there exist a Kasparov A, IB-module $\mathcal{G} \in \mathbb{E}(A, IB)$ such that $G_{\pi_0} \cong \mathcal{E}$ and $\mathcal{G}_{\pi_1} \cong \mathcal{F}$. We write $\mathcal{E} \sim_h \mathcal{F}$ if there is a finite set $\{\mathcal{E}_1, \dots, \mathcal{E}_n\} \in \mathbb{E}(A, B)$ of Kasparov A, B-modules, such that $\mathcal{E}_1 = \mathcal{E}$, $\mathcal{E}_n = \mathcal{F}$ and $\mathcal{E}_i$ is homotopic to $\mathcal{E}_{i+1}$ for $i = 1, \dots, n-1$.

We obviously now want to prove that $\sim_h$ is an equivalence relation. To do this we want to introduce the *internal tensor product* of Kasparov modules.

Construction 2.2.6 (Internal tensor product): Let $\mathcal{E} = (E, \phi, F) \in \mathbb{E}(A, B)$ and $\psi\colon B \to C$ be a graded *-homomorphism. We can form a Hilbert C-module $E \otimes_\psi C$. By construction we can define a grading operator $S_{E\otimes_\psi C}$ on $E \otimes_\psi C$, by defining

$$S_{E\otimes_\psi C}(e \otimes_\psi c) = S_E(e) \otimes_\psi \beta_C(c)$$

for $e \in E$, $c \in C$.

Note that as in Construction 2.1.15, we have a *-homomorphism j such that $j(m) = m \otimes_\psi \mathrm{id}$ for $m \in \mathcal{B}_B(E)$. Thus we can define $\phi \otimes_\psi \mathrm{id}\colon A \to \mathcal{B}_C(E \otimes_\psi C)$ by $\phi \otimes_\psi \mathrm{id}(a) = \phi(a) \otimes_\psi \mathrm{id} \in \mathcal{B}_C(E \otimes_\psi C)$ for $a \in A$. Moreover by Lemma 2.1.16, we have that $m \otimes_\psi \in \mathcal{K}_B(E \otimes_\psi C)$ if $m \in \mathcal{K}_B(E)$. By this we obtain that

$$(E \otimes_\psi C, \phi \otimes_\psi \mathrm{id}, F \otimes_\psi \mathrm{id}) \in \mathbb{E}(A, B),$$

which we will denote by $\psi_*(\mathcal{E})$.

Moreover if E is countably generated then also $E \otimes_\psi C$ is countably generated.

To prove now that $\sim_h$ is an equivalence relation we still need two technical lemmas. For a proof see for instance [JT91].

Lemma 2.2.7: *Let $\mathcal{E} \in \mathbb{E}(A, B)$ and $f\colon B \to C$ and $g\colon C \to D$ be surjective graded *-homomorphisms. Then*

$$g_*(f_*(\mathcal{E})) \cong (g \circ f)_*(\mathcal{E}).$$

Lemma 2.2.8: *Let $\mathcal{E} \in \mathbb{E}(A, B)$ and $f\colon B \to C$ be a surjective graded *-homomorphism. Then*

$$\mathcal{E}_f \cong f_*(\mathcal{E}).$$

Now we have everything to prove that $\sim_h$ is indeed an equivalence relation on the set of Kasparov modules.

Proposition 2.2.9: *The relation $\sim_h$ is an equivalence relation on $\mathbb{E}(A, B)$.*

Proof. One only needs to check, that $\sim_h$ is reflexive and symmetric. By construction transitivity is clear. First let $\mathcal{E} \in \mathbb{E}(A, B)$. Let $\phi\colon B \to IB$, $b \mapsto (t \mapsto b)$, then $\pi_t \circ \psi = \mathrm{id}$ for all $t \in [0, 1]$. Hence $\psi_*(\mathcal{E}) \in \mathbb{E}(A, IB)$ and moreover by Lemma 2.2.7 and Lemma 2.2.8 we have

$$\psi_*(\mathcal{E})_{\pi_0} \cong (\pi_0 \circ \psi))_*(\mathcal{E}) = \mathrm{id}_*(\mathcal{E}) \cong \mathcal{E}.$$

The same holds for π_1. Thus $\mathcal{E} \sim_h \mathcal{E}$, and this proves reflexivity of $\sim_h$.

Secondly let $\mathcal{E}, \mathcal{F} \in \mathbb{E}(A, B)$ such that $\mathcal{E} \sim_h \mathcal{F}$. Using the identification $B \otimes C([0, 1]) \cong C([0, 1], B)$ the idea is to inverse the setting in some sense. To do so let $\psi'\colon C([0, 1]) \to C([0, 1])$, $(t \mapsto f(t)) \mapsto (t \mapsto f(1 - t))$, which is a *-isomorphism and define $\psi = \mathrm{id} \otimes \psi'$ which is by construction also a *-isomorphism. Moreover we have by construction that $\pi_0 \circ \psi = \pi_1$ and $\pi_1 \circ \psi = \pi_0$. Since there exist $\mathcal{G} \in \mathbb{E}(A, IB)$ such $\mathcal{G}_{\pi_0} \cong \mathcal{E}$ and $\mathcal{G}_{\pi_1} \cong \mathcal{F}$, we obtain again by applying Lemma 2.2.7 and Lemma 2.2.8, that

$$\psi_*(\mathcal{G})_{\pi_0} \cong (\pi_0 \circ \psi)_*(\mathcal{G}) \cong (\pi_1)_*(\mathcal{G}) \cong \mathcal{F}.$$

Analogously we obtain $\psi_*(\mathcal{G})_{\pi_1} \cong \mathcal{E}$. □

Since $\sim_h$ is an equivalence relation, we now can define the KK-groups. But let us mention that there are also other equivalence relations such as *operator homotopy* (Definition 2.2.12), and others, which we can define on $\mathbb{E}(A, B)$, see for instance [Bla98]. To avoid the different "induced KK-theories", from now on the first argument of the set of Kasparov modules is assumed to be separable and the second argument should be σ-unital. The last assumptions can be explained roughly by the fact, that otherwise B may not have "enough" countably generated Hilbert C^*-modules.

Definition 2.2.10 (The KK-groups): We set

$$KK(A, B) := KK^0(A, B) := \mathbb{E}(A, B)/ \sim_h$$

and define

$$KK^1(A, B) := KK^1(A, B) = KK(A, B \hat{\otimes} \mathbb{C}_{(1)}),$$

where $\mathbb{C}_{(1)}$ is defined as in Example 2.1.3 and $\hat{\otimes}$ is the minimal graded tensor product of C^*-algebras. The elements, i.e. the equivalence classes, of $KK(A, B)$ respectively $KK^1(A, B)$ are denoted by $[\mathcal{E}]$.

The definition in [Kas80] is slightly different, since Kasparov also divided out the set $\mathbb{D}(A, B)$, but this is not necessary, since if $\mathcal{E} = (E, \phi, F) \in \mathbb{D}(A, B)$, then $\mathcal{E}$ is homotopic to 0. See for instance [JT91].

We want to equip $KK(A, B)$ with an addition, which is obviously via the direct sum of Kasparov modules. Moreover it is commutative.

Definition 2.2.11 (Direct sum as group operation): For $[\mathcal{E}], [\mathcal{F}] \in \mathbb{E}(A, B)$, we set

$$[\mathcal{E}] + [\mathcal{F}] = [\mathcal{E} \oplus \mathcal{F}].$$

We now can prove that $KK(A, B)$ is an abelian group, but to do so let us introduce the mentioned *operator homotopy*, which gives us the same KK-groups by assumption.

Definition 2.2.12 (Operator homotopy): Let $\mathcal{E}, \mathcal{F}$ be Kasparov A, B-modules. If there is a graded Hilbert B-module E, a graded *-homomorphism $\phi\colon A \to \mathcal{B}_B(E)$ and a norm continuous path G_t for $t \in [0, 1]$ such that

- $\mathcal{G}_t = (E, \phi, G_t)$ is a Kasparov A, B-module,
- $\mathcal{G}_0 \cong \mathcal{E}$ and $\mathcal{G}_1 \cong \mathcal{F}$,

then $\mathcal{E}$ and $\mathcal{F}$ are called *operator homotopic.*

Proposition 2.2.13: $(KK(A, B), +)$ *is an abelian group.*

Proof. As identity $[0]$ we simply take degenerate Kasparov A, B-modules, since all degenerate Kasparov A, B-modules are homotopic to 0.

Associativity is clear, so we need to construct an inverse for an element. Let $\mathcal{E} = (E, \phi, F)$ and define E^{op} as the Hilbert B-module E graded with $-S_E$, and define $\phi^{op} \colon A \to \mathcal{B}_B(E^{op}) = \mathcal{B}_B(E)$ by $\phi^{op} = \phi \circ \beta_A$, where β_A is the grading automorphism of A. Note that ϕ^{op} is a graded *-homomorphism. We now prove that $-\mathcal{E} := (E^{op}, \phi^{op}, -F)$ is the inverse. To do so let

$$G_t = \begin{pmatrix} F\cos(\frac{\pi}{2}t) & \mathrm{id}\sin(\frac{\pi}{2}t) \\ \mathrm{id}\sin(\frac{\pi}{2}t) & -F\cos(\frac{\pi}{2}t) \end{pmatrix},$$

for $t \in [0, 1]$. Note that $G_0 = F \oplus -F$ and G_1, hence $(E \oplus E^{op}, \phi \oplus \phi^{op}, F \oplus -F) = (E \oplus E^{op}, \phi \oplus \phi^{op}, G_0)$. Moreover G_1 is odd (see Example 2.1.3) since

$$G_1 = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}.$$

Computations as in [JT91] prove that $(E \oplus E^{op}, \phi \oplus \phi^{op}, G_t)$ is a Kasparov A, B-module for all $t \in [0, 1]$, moreover setting $t = 1$ yields us that $(E \oplus E^{op}, \phi \oplus \phi^{op}, G_1)$ is a degenerate Kasparov module, since $G_1 = G_1^*$ and $G_1^2 = I_2$. Hence we have a operator homotopy between $\mathcal{E} \oplus -\mathcal{E}$ and a degenerate Kasparov module. Since homotopy and operator homotopy gives us the same KK-groups, we are done. ☐

We now want to collect the properties of the "bifunctor" KK. Note that the most proofs are technical, thus we do not mention them.

Proposition 2.2.14: (i) $KK(-, B)$ *is a contravariant functor from the category of separable and graded C^*-algebras to the category of abelian groups via the pullback ψ^*.*

(ii) $KK(A, -)$ *is a covariant functor from the category of σ-unital graded C^*-algebras to the category of abelian groups via the internal tensor product ψ_*.*

(iii) $KK(-, B)$ *and* $KK(A, -)$ *are "homotopy invariant", i.e. for a path of *-homomorphisms $\psi_t \colon A \to B$ with $t \mapsto \psi_t(a)$ continuous for all $a \in A$ we have $\psi_0^* = \psi_1^*$ respectively $(\psi_0)_* = (\psi_1)_*$.*

(iv) *By Kasparov stabilisation theorem (see [Bla98]) we only need to consider those Kasparov A, B-modules (E, ϕ, F) such that $E = \mathbb{H}_B$.*

(v) *For a Kasparov A, B-module (E, ϕ, F) one may assume $F = F^*$ and $\|F\| \leq 1$, hence we only need to consider compact perturbations.*

(vi) *Let B be trivially graded then we have the following isomorphisms*

$$KK^0(\mathbb{C}, B) \cong K_0(B),$$
$$KK^1(\mathbb{C}, B) \cong K_1(B).$$

Remark 2.2.15: (i) This last proposition gives an understanding why we can consider KK-theory as a bifunctorial generalisation of classical K-theory. The same holds for example for the *Bott periodicity*, since we can define KK^n, but as in the case of K-theory it turns out, that one only needs to consider KK^0 and KK^1. However the proof in the general setting of KK-theory is easier.

(ii) Moreover we should mention that there are different "pictures" or ways one can think about KK-theory, such as the *Fredholm picture* or the *Cuntz picture* using so called quasi homomorphisms.

It is normally much harder to compute the KK-theory of two given C^*-algebras. One tool may be the so called *Kasparov product*, which can be elaborately constructed. We will omit any proof since everything needs a highly amount of technical arguments. As notation for $[(B, \phi, 0)] \in KK(A, B)$ we write $[\phi]$.

Proposition 2.2.16 (Kasparov product): *Let A, B, C, D be σ-unital graded C^*-algebras, then there exists a map, the so called Kasparov product*

$$\otimes_B \colon KK(A, B) \times KK(B, C) \to KK(A, C),$$

that has the following properties:

- *biadditivity with respect to $\oplus$, i.e.*

$$(x \oplus y) \otimes_B z = x \otimes_B z \oplus y \otimes_B z,$$
$$z \otimes_B (x \oplus y) = z \otimes_B x \oplus_B z \otimes y,$$

- *associativity,*
- *unit elements $1_A = [\mathrm{id}_A] \in KK(A, B)$ and $1_B = [\mathrm{id}_B] \in KK(B, C)$ such that*

$$1_A \otimes_B x = x = x \otimes 1_B,$$

- *functoriality: if $\phi \colon A \to B$ and $\psi \colon B \to C$ are graded *-homomorphisms, then*

$$[\phi] \otimes_B x = \phi_*(x) \in KK(A, C),$$
$$x \otimes_B [\psi] = \psi_*(x) \in KK(A, C).$$

The main problem about the proof is to explicitly construct the rank one operator. However one can quickly construct the Hilbert module and the graded *-homomorphism by simply taking the graded tensor product.

Let us quickly view on the category-theoretical conclusions of the Kasparov product, mentioned in [Par09].

Remark 2.2.17: Taking separable graded C^*-algebras we can form a *additive category*, where we take KK-groups as morphism sets and the flipped Kasparov product as compositions. The map $\psi \mapsto [\psi]$ is a functor from the category of separable C^*-algebras with graded *-homomorphisms in this additive category.

The isomorphisms in this category are called *KK-equivalence.*

The notion of KK-equivalence is a powerful tool in a lot of various contexts in operator algebras.

Definition 2.2.18 (KK-equivalence): An element $x \in KK(A, B)$ is called *KK-equivalence* if there exists an element $y \in KK(B, A)$ with $x \otimes_B y = 1_A$ and $y \otimes_A x = 1_B$.

We say that A and B are *KK-equivalent* if there exist a KK-equivalence. Often we say that y is the inverse of x, if x is a KK-equivalence.

Note that if $x \in KK(A, B)$ is a KK-equivalence, then for any separable C^*-algebra D the maps $x \otimes_B - \colon KK(B, D) \to KK(A, D)$ and $- \otimes_B x \colon KK(D, A) \to KK(D, B)$ are isomorphisms, as it is explained in [Bla98]. Hence the KK-theory of KK-equivalent C^*-algebras behave "identically". Moreover KK-equivalence implies the same K-theory.

If one assume additional assumptions for two separable trivially graded C^*-algebras A, B. Then one can say that A and B are KK-equivalent if and only if their K-theory is equal, the so called *Universal Coefficient Theorem*, [Bla98].

Chapter III.

Compact quantum groups and their representation theory

The notion of compact quantum groups was introduced by Woronowicz in [Wor87; Wor98] to generalise Pontryagin duality of abelian compact groups in some sense. In this chapter we want to introduce the basic definitions, look at some class of examples, and we want to look at the representation theory of compact quantum groups to then introduce the dual discrete quantum group. We will conclude with introducing the free wreath product of compact quantum groups with S_N^+.

The symbol $\otimes$ will denote the minimal tensor product of C^*-algebras. Moreover following the literature we will omit $\circ$ for the chaining of functions, if it is obvious from the context, what we mean.

1. The category of compact quantum groups

In this section we define compact quantum groups and some notions, we will use in this thesis. One can see compact quantum groups as a kind of generalisation of (locally) compact groups, which will be motivated in the following.

Let G be a compact group with an operation $\circ\colon G \times G \to G$. Dualising this operation, we obtain a map

$$\Delta\colon C(G) \to C(G \times G) \cong C(G) \otimes C(G),\ f \mapsto ((g,h) \mapsto f(gh))$$

Mainly following the philosophy of Proposition 1.1.8, we may now replace $C(G)$ by a non-commutative C^*-algebra. Following this idea our C^*-algebra should have a map Δ as above, with additional properties.

Definition 3.1.1 (Compact Quantum Group): A *compact quantum group* G is a unital C^*-algebra $C(G)$ together with a unital *-homomorphism (*comultiplication*) $\Delta\colon C(G) \to C(G) \otimes C(G)$ such that

- $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ (*coassociativity*)

- $[(C(G) \otimes 1)\Delta(C(G))] = [(1 \otimes C(G))\Delta(C(G))] = C(G) \otimes C(G)$
  (*cancellation law*),

where $[\cdot]$ denotes the closed linear span.

The first following example is a kind of justification why we can see compact quantum groups as a generalisation of the continuous functions on a compact group. Moreover the second example is the firsz historical example, given by Woronowicz, and the third one should be seen as an example coming from the dual, that we will consider later in the course of this chapter.

Example 3.1.2: (i) Let G be a compact group. Consider $C(G)$ with Δ constructed as above. It is easy to prove that Δ is indeed coassociative and also the cancellation law is true. Hence $(C(G), \Delta)$ is a commutative compact quantum group.

By the Gelfand-Naimark theorem every commutative compact quantum group is of the form $C(G)$. This justifies the notation. In general for non-commutative compact quantum groups, our underlying C^*-algebra is obviously not of the form $C(G)$, but we keep the notation in the spirit of Gelfand-Naimark, and we view $C(G)$ as a kind of "virtual" continuous function space.

(ii) Historically, the first example of Woronowicz was the quantum version of the special unitary group $SU(2)$. To do so he *deformed* the commutativity.

Note that $SU(2)$ can be written as

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\gamma} \\ \gamma & \overline{\alpha} \end{pmatrix} \middle| \alpha, \gamma \in \mathbb{C},\ |\alpha|^2 + |\gamma|^2 = 1 \right\}.$$

Now let $q \in [-1, 1]\backslash\{0\}$, then define the deformation via the universal C^*-algebra

$$C(SU_q(2)) = C^* \left(1, \alpha, \gamma \middle| \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is unitary} \right).$$

One then equip $SU_q(2)$ with a comultiplication Δ defined on the generators

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma,\ \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

(iii) Let Γ be a discrete group, i.e. equipped with the discrete topology. We consider the reduced group C^*-algebra $C_r^*(\Gamma)$ which we obtain by taking norm closure of the group ring $\mathbb{C}[\Gamma]$ under the regular representation $\lambda\colon \Gamma \to B(\ell^2(\Gamma))$ mapping g to λ_g given by $\lambda_s(\delta_t) = \delta_{st}$ on the orthonormal basis on $\ell^2(\Gamma)$. Define the comultiplication via $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$. This defines us a compact quantum group $(C_r^*(\Gamma), \Delta)$ which is moreover cocommutative, i.e. $\sigma\Delta = \Delta$, where σ denotes the flip operator on the tensor product mapping $a \otimes b$ to $b \otimes a$.

By taking the closure under the universal norm of the group algebra $\mathbb{C}[\Gamma]$ we obtain the full group C^*-algebra $C^*(\Gamma)$. Then $(C^*(\Gamma), \Delta)$ is also a cocommutative compact quantum group.

Before defining homomorphisms of compact quantum groups and compact quantum subgroups, we want to fix the following remark, regarding our example.

Remark 3.1.3: For a discrete group Γ, we denote $\widehat{\Gamma}$ for either $(C^*(\Gamma), \Delta)$ or $(C_r^*(\Gamma), \Delta)$, where $\Delta(\lambda_g) = \lambda_g \otimes \lambda_g$ and call $\widehat{\Gamma}$ the *dual* of Γ.

Indeed this construction generalises the *Pontryagin duality* for compact abelian groups. We will take a closer look at the concept of duality in the course of this chapter.

Lastly note that every cocommutative compact quantum group $G = (C(G), \Delta')$, can be included between the full group C^*-algebra and the reduced group C^*-algebra of a suitable discrete group Γ, i.e. we have surjective *-homomorphisms

$$C^*(\Gamma) \to C(G) \to C_r^*(\Gamma)$$

intertwining the comultiplications. See for instance [Tim08].

In our example we can understand $C(G)$ and $C_r^*(\Gamma)$ as a *compact quantum subgroup* of $C^*(\Gamma)$ respectively $C(G)$, as we will define now via the notion of homomorphisms of compact quantum groups.

Definition 3.1.4 (CQG homomorphism): Let $G = (C(G), \Delta_G)$ and $H = (C(H), \Delta_H)$ be compact quantum groups. A *compact quantum group homomorphism* from G to H is a unital *-homomorphism $\pi\colon C(G) \to C(H)$ such that

$$(\pi \otimes \pi)\Delta_G = \Delta_H \pi.$$

We also say that π *intertwines the comultiplications.*

Definition 3.1.5 (Compact quantum subgroup): Let $G = (C(G), \Delta_G)$ be a compact quantum group. We call $H = (C(H), \Delta_H)$ a *compact quantum subgroup* of G, if there is a surjective homomorphism of compact quantum groups from G to H.

For a compact group G we get by Riesz's theorem the existence of a unique "left-invariant" *Haar measure.* In the case of compact quantum groups we get an analogue of a Haar measure. This was firstly proven by Woronowicz [Wor87; Wor98].

Proposition 3.1.6 (Existence of a Haar state): *Let G be a compact quantum group with comultiplication Δ. Then there is a unique state $\phi\colon C(G) \to \mathbb{C}$ such that*

$$(\mathrm{id} \otimes \phi)\Delta(f) = \phi(f)1 = (\phi \otimes \mathrm{id})\Delta(f)$$

for all $f \in C(G)$. We say that ϕ is left- and right-invariant. We call ϕ Haar state of G.

We want to split this proof in smaller lemmas.

Lemma 3.1.7: *Let G be a compact quantum group. Let ϕ, ψ be states on $C(G)$ and $a \in C(G)$, then*

$$\phi((\psi \otimes \mathrm{id})\Delta(a)) = \psi((\mathrm{id} \otimes \phi)\Delta(a))$$

Proof. Let $\Delta(a) = \sum_i a_1^i \otimes a_2^i$, then

$$\phi((\psi \otimes \mathrm{id})\Delta(a)) = \sum_i \phi(\psi(a_1^i)a_2^i) = \sum_i \psi(a_1^i \phi(a_2^i)) = \psi((\mathrm{id} \otimes \phi)\Delta(a)). \qquad \square$$

Fix for two states ϕ, ψ the notation $\phi * \psi = (\phi \otimes \psi)\Delta$.

Lemma 3.1.8: *Let G be a compact quantum group and ω be a state on $C(G)$, then there exists a state ϕ such that $\omega\phi = \phi\omega = \phi$.*

Proof. Let w_n be the Césaro sum

$$\omega_n = \frac{1}{n} \sum_{k=1}^{n} \omega^{*k},$$

where we denote inductively $\omega^{*k+1} = \omega * \omega^{*k}$, $\omega^{*1} = \omega$. Since the set of states of any unital C^*-algebra is compact with respect to the weak topology, we have a weak accumulation point ϕ by Banach-Alaoglu. We then get

$$\omega_n * \omega = \omega * \omega_n = \omega_n + \frac{1}{n}(\omega^{*(n+1)} - \omega),$$

and therefore $\phi * \omega = \omega * \phi = \phi$. $\square$

Proof (of Proposition 3.1.6). The uniqueness follows directly from Lemma 3.1.7, since by assumption the Haar state is left and right invariant.

Now define for a state $\omega \in C(G)^*$ (continuous linear functionals on $C(G)$)

$$K_\omega := \{\rho \in C(G)^* \mid \rho \text{ state}, \ \rho * \omega = \omega * \rho = \omega(1)\rho\}.$$

By Lemma 3.1.8 $K_\omega \neq \emptyset$.

Now let $n \in \mathbb{N}$ be arbitrary and $\omega_1, \ldots, \omega_n$ be states on $C(G)$, we want to show that $\cap_{i=1}^n K_{\omega_i}$ is not empty. To do so, define $\omega := \omega_1 + \omega_2$. We need to prove $K_\omega \subseteq K_{\omega_1}$, then $\emptyset \neq K_\omega \subseteq K_{\omega_1} \cap K_{\omega_2}$ and inductively $K_{\sum_{i=1}^n \omega_i} \subseteq \cap_{i=1}^n K_{\omega_i}$.

For this let $\rho \in K_\omega$. Define the left ideal

$$L_{\rho \otimes \omega} = \{q \in C(G) \otimes C(G) \mid (\rho \otimes \omega)(q^* q) = 0\} \subseteq C(G).$$

By definition, we obviously get $L_{\rho\otimes\omega} \subseteq L_{\rho\otimes\omega_1}$. Using Cauchy-Schwarz ([LVW20, Lemma 5.4.]), we obtain $L_{\rho\otimes\omega_1} \subseteq \ker(\rho \otimes \omega_1)$. Defining $\Psi \colon C(G) \to C(G)$ by

$\Psi(x) = (\mathrm{id} \otimes \rho)\Delta(x) - \rho(x)1$. One then can show, that $(\mathrm{id} \otimes \Psi)\Delta(C(G)) \subseteq L_{\rho\otimes\omega}$. Moreover we get the following inclusions

$$1 \otimes \Psi(C(G)) \subseteq (C(G) \otimes 1)L_{\rho\otimes\omega} \subseteq \ker(\rho \otimes \omega_1).$$

By this we can conclude for $x \in C(G)$

$$0 = (\rho \otimes \omega_1)(1 \otimes \Psi(x)) = (\omega_1 \otimes \rho)\Delta(x) - \omega_1(1)\rho(x)$$

and thus $\rho \in K_{\omega_1}$.

By Cantor's intersection principle there exist $\phi \in \bigcap_{\omega \text{ state}} K_\omega$, such that we have $\phi * \omega = \omega * \phi = \phi$ for all states ω. For $x \in C(G)$, let $y = \phi(x) - (\mathrm{id} \otimes \phi)\Delta(x)$. By construction $\omega(y) = 0$ for all states ω, i.e. $y = 0$. □

Definition 3.1.9 (Kac type): A compact quantum group G is said to be of *Kac type* if its (unique) Haar state h is a trace, i.e. if for all $x, y \in C(G)$ we have $h(xy) = h(yx)$.

One should note that often Kac type is defined via the *antipode* S of the corresponding Hopf-*-algebra, see Definition 3.3.12. We say equivalently that a compact quantum group is of Kac-type, if $S^2 = \mathrm{id}$.

We now want to define the so called *reduced C^*-algebra of functions on G* and in the course of this chapter the *full C^*-algebra of functions on G*.

Definition 3.1.10 (Reduced C^*-algebra of functions on G): Let G be a compact quantum group, and denote by h its Haar state. The image of $C(G)$ in the GNS-representation π_h (Proposition 1.1.11) is denoted by $C_r(G)$ and is called the *reduced C^*-algebra of functions on G*.

Since $(\pi_h \otimes \pi_h)\Delta$ can be factorised through π_h one obtain a comultiplication $\Delta_r \colon C_r(G) \to C_r(G) \otimes C_r(G)$ such that

$$\Delta_r \pi_h = (\pi_h \otimes \pi_h)\Delta.$$

By this $(C_r(G), \Delta_r)$ has a natural structure of a compact quantum group. Moreover the reduced one is a compact quantum subgroup of G.

2. Compact matrix quantum groups

In this section we want to define compact matrix quantum groups, and by this a large class of examples for compact quantum groups. In Corollary 3.3.8 we will see that compact matrix quantum groups are indeed compact quantum groups. Moreover we will look mostly at some classical examples.

Let N be a natural number.

Definition 3.2.1 (Compact matrix quantum group): Let $C(G)$ be a unital C^*-algebra which is generated by the entries of a matrix $u \in M_N(C(G))$, where u and u^* are invertible. Moreover assume that

$$\Delta\colon C(G) \to C(G) \otimes C(G), \Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}$$

is a *-homomorphism. Then the tuple $G = (C(G), u)$ is called *compact matrix quantum group.* The matrix u is called the *fundamental representation* of the compact matrix quantum group.

The following three examples of compact matrix quantum groups were introduced in [Wan95] and [Wan98].

Definition 3.2.2 (Quantum symmetric group): The *quantum symmetric group* $S_N^+ = (C(S_N^+), u)$ is defined by

$$C(S_N^+) = C^*(u_{ij} \mid u_{ij}^* = u_{ij}^2 = u_{ij}, \sum_i u_{ik} = \sum_i u_{ki} = 1 \text{ for all } k \leq N)$$

Such a matrix $u = (u_{ij})$ is called *magic unitary.* The quantum subgroups of S_N^+ are called *quantum permutation groups.*

The following remark will explain why we can see S_N^+ as some kind of generalisation of the symmetric group S_N or more precisely the continuous functions on S_N. Also we want to look for which natural number N the quantum symmetric group is non-commutative.

Remark 3.2.3: (i) For $N \geq 4$ the C^*-algebra $C(S_N^+)$ is non-commutative. Indeed let $N = 4$ and B be a C^*-algebra generated by two non-commuting projections p, q. Due to the universal property of $C(S_4^+)$ we obtain a *-homomorphism which maps u_{ij} to the matching entry of the following matrix

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}.$$

Since B is non-commutative $C(S_4^+)$ is also non-commutative, hence we obtain more general $S_N \ncong S_N^+$ for $N \geq 4$, but $S_N \subseteq S_N^+$ as compact quantum groups.

For the case $N \leq 2$, it is easy to show that S_N^+ is commutative. Indeed let $N = 1$, then $C(S_1^+) = \mathbb{C}$, and for $N = 2$ the fundamental representation is of the form

$$\begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix},$$

where p is some projection and obviously all generators commute. For $N = 3$ the quantum symmetric group is also commutative, see for instance [Web23]. One needs to use the fact, that if projections sum up to 1 they need to be mutually orthogonal.

(ii) One can show that the abelisation of $C(S_N^+)$, i.e. taking the quotient with the two-sided closed ideal generated by $u_{ij}u_{kl} - u_{kl}u_{ij}$ is isomorphic to $C(S_N)$, so S_N^+ can be indeed understood as a generalisation of S_N.

It is not easy to determine the Haar state of S_N^+ for words of length ≥ 2, which can be done using the *Gram-Weingarten-formula*, [BC07]. For the generators we can easily find, that all matrix coefficients have the same weight.

Lemma 3.2.4: *Let $N \geq 1$, and let $u = (u_{ij})$ be the fundamental matrix of S_N^+. Denote by h the Haar state of S_N^+, then*

$$h(u_{ij}) = \frac{1}{N}.$$

Proof. We have for all $1 \leq i \leq N$ by linearity of h, that

$$h(1) = h(\sum_j u_{ij}) = \sum_j h(u_{ij}) = 1.$$

Since we can permutate the matrix coefficients u_{ij}, we obtain that the weight of every matrix coefficient must be $\frac{1}{N}$. □

Definition 3.2.5 (Orthogonal quantum group): For $N \in \mathbb{N}$ the *free orthogonal quantum group* $O_N^+ = (C(O_N^+), u)$ is defined by

$$C(O_N^+) = C^*(u_{ij} \mid u_{ij}^* = u_{ij},\ u \text{ orthogonal}).$$

Remark 3.2.6 (O_N^+ as quantum isometry group of non-commutative sphere): As proven in [BG10] the classical sphere $S^{n-1} \subseteq \mathbb{R}^n$ can be seen as the spectrum of the universal C^*-algebra

$$C^*(x_1, \ldots, x_n \mid x_i = x_i^*,\ x_i x_j = x_j x_i,\ \sum_i x_i^2 = 1).$$

We can also express the orthogonality of u by the following relations

$$\sum_k u_{ik}u_{jk} = \delta_{ij}.$$

By this O_N^+ gives us a kind of a *non-commutative sphere.*

Definition 3.2.7 (Unitary quantum group): Let $N \in \mathbb{N}$. The *free unitary quantum group* $U_N^+ = (C(U_N^+), u)$ is defined by

$$C(U_N^+) = C^*(u_{ij} \mid u, u^T \text{ unitary}).$$

Finally we define the *free hyperoctahedral quantum group*, which was introduced by Bichon in [Bic04].

Definition 3.2.8 (Hyperoctahedral quantum group): The *free hyperoctahedral quantum group* $H_N^+ = (C(H_N^+), u)$ is defined by

$$C(H_N^+) = C^*(u_{ij} \mid u_{ij} = u_{ij}^*,\ u \text{ orthogonal},\ u_{ik}u_{jk} = 0 = u_{ki}u_{kj} \text{ for } i \neq j).$$

Moreover we could also define H_N^+ as the *free wreath product* $H_N^+ = \widehat{\mathbb{Z}/2\mathbb{Z}} \wr_* S_N^+$, which we will see later. And by this we also obtain for $N \geq 2$, that H_N^+ is non-commutative, since we can embed S_4^+ in H_2^+.

3. Representation theory of compact quantum groups

In this section we want to define basic notions of representation theory of compact quantum groups. To motivate the following definition, we want to have a look at compact groups and their representation theory.

Useful in this section is the notation of *leg numbering* for C^*-algebras. We use for an element a in $A \otimes A$ the notation $a_{(12)}, a_{(13)}, a_{(23)}$ for the elements obtained by the inclusion of $A \otimes A$ in $A \otimes A \otimes A$, e.g. $a_{(12)} = a \otimes 1$.

Motivation 3.3.1: Let G be a compact group and $U \colon G \to M_n(\mathbb{C})$ be a finite-dimensional representation.

Note that U is continuous by definition, i.e. $U \in C(G, M_n(\mathbb{C})) \cong C(G) \otimes M_n(\mathbb{C})$. By this we can also write $U = \sum_{i,j} u_{ij} \otimes e_{ij}$ for some u_{ij}, where e_{ij} denotes the (i,j)-matrix unit. Moreover since $U(gh) = U(g)U(h)$, we obtain

$$\begin{aligned} U(gh) &= \sum_{i,j} u_{ij}(gh) \otimes e_{ij} = \sum_{i,j} \Delta(u_{ij})(g,h) \otimes e_{ij} \\ &= \sum_{i,j} \left(\sum_k u_{ik}(g) u_{kj}(h) \right) \otimes e_{ij} = U(g)U(h), \end{aligned}$$

where Δ is defined as in the previous section. Therefore by comparing, we get

$$\Delta(u_{ij})(g,h) = \sum_k u_{ik}(g)u_{kj}(h) = \left(\sum_k u_{ik} \otimes u_{kj} \right)(g,h).$$

This motivates the following definition in the finite-dimensional case.

Definition 3.3.2 (Representation of CQG): Let G be a compact quantum group.

(i) A *representation* of G on a Hilbert space $\mathcal{H}_V$ is an element V in the multiplier algebra $M(\mathcal{K}(\mathcal{H}_V) \otimes C(G))$ such that

$$(\mathrm{id} \otimes \Delta)(V) = V_{(12)}V_{(13)}.$$

If V is unitary then we say V is a unitary representation.

(ii) A *representation of degree* n is a matrix $U = (u_{ij}) \in M_n(C(G))$ such that for all $1 \leq i, j \leq n$, we have

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

A representation is called *unitary* if $U = (u_{ij})$ is a unitary. We say that U is *non-degenerated* if U is invertible.

(iii) A linear map $T\colon \mathcal{H}_U \to \mathcal{H}_V$ is called a *intertwiner* between the representation U on $\mathcal{H}_U$ and the representation V on $\mathcal{H}_V$, if

$$(T \otimes \mathrm{id})U = V(T \otimes \mathrm{id}).$$

We denote the set of intertwiners between U and V by $\mathrm{Mor}(U, V)$.

(iv) If there is a unitary intertwiner in $\mathrm{Mor}(U, V)$, then U and V are called *(unitarily) equivalent.*

(v) A unitary representation U is called *irreducible* if $\mathrm{Mor}(U, U) = \mathbb{C}1$. Denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible unitary representations of G.

As shown in [Wor87] any irreducible representation is finite dimensional. Because of this and for technical reasons, we will mostly restrict our self to finite-dimensional representations. Moreover note that the definition for general representations coincide with the one for finite-dimensional representations in the case of finite-dimensionality.

We can also define operations on representations, namely the direct sum and tensor product. For the following definition note that $M_n(\mathbb{C}) \otimes C(G)$ is isomorphic to $M_n(C(G))$.

Definition 3.3.3: Let G be a compact quantum group, and let U, V be representations of dimension n and m.

(i) The direct sum $U \oplus V$ is defined as the element of $M_{n+m}(C(G))$ obtained as the diagonal sum of the two representations.

(ii) The tensor product $U \otimes V$ is the element

$$U \otimes V = U_{(13)}V_{(23)} \in M_{mn}(C(G))$$

In analogy to the classical Peter-Weyl representation theory for groups, we obtain completely analogue results. A useful lemma is the one stated in [MV98, Lemma 6.3], which we will use multiple times. For the rest of the chapter denote h for the Haar state of some compact quantum group G.

Lemma 3.3.4: *Let V and W be representations of G on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. For a compact operator $x \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ define*

$$y = (\mathrm{id} \otimes h)(W^*(x \otimes 1)V).$$

Then $y \in \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ *such that*

$$W^*(y \otimes 1)V = y \otimes 1.$$

Proposition 3.3.5: *Every non-degenerate irreducible finite-dimensional representation of a compact quantum group is equivalent to a unitary one. Furthermore every unitary finite-dimensional representation can be decomposed as a direct sum of irreducible representations.*

Proof. Let U be a non-degenerate representation of degree n and define

$$y = (\mathrm{id} \otimes h)(U^*U).$$

Since U is invertible also U^*U is invertible, and moreover as a positive and invertible element, U^*U is strictly positive, therefore there exist $\varepsilon > 0$ such that $U^*U \geq \varepsilon 1$. Note that h as Haar state is positive, thus $y \geq 0$ and $y \geq \epsilon 1$, i.e. y is invertible. Now let

$$W = (y^{\frac{1}{2}} \otimes 1)U(y^{-\frac{1}{2}} \otimes 1)$$

By Lemma 3.3.4 we have

$$y \otimes 1 = U^*(y \otimes 1)U$$

and by this we obtain

$$\begin{aligned} W^*W &= (y^{-\frac{1}{2}} \otimes 1)U^*(y^{1/2} \otimes 1)(y^{1/2} \otimes 1)U(y^{-\frac{1}{2}} \otimes 1) \\ &= (y^{-\frac{1}{2}} \otimes 1)U^*(y \otimes 1)U(y^{-\frac{1}{2}} \otimes 1) \\ &= (y^{-\frac{1}{2}} \otimes 1)(y \otimes 1)(y^{-\frac{1}{2}} \otimes 1) = 1. \end{aligned}$$

Thus W is unitary and U and W are unitary equivalent.

For the second part of the statement, let V be a unitary finite-dimensional representation. Since we are finite-dimensional, the unitary finite-dimensional representation is decomposable as a direct sum of (not necessarily irreducible) unitary sub-representations. By finite-dimensionality decomposing these elements of the direct sum, must lead us to irreducible unitary representations. □

Remark 3.3.6: The last proof works also in a more general setting, since we can drop the assumption of finite-dimensionality. To do so, in particular for the second part, we prove that the set of intertwiners acts non-degeneratively on the set of bounded operators on the Hilbert space on which we have the representation. Then one takes a maximal family of orthogonal minimal projections on the set of intertwiners. By using that the set of intertwiners acts non-degeneratively. one obtain the statement. See for instance [MV98].

We want to prove the following important statement, which will us help to understand how C^*-algebraic quantum groups and Hopf *-algebras /algebraic quantum groups are connected.

Proposition 3.3.7 (Generation by coefficients of representations): *Let A be a unital C^*-algebra with a *-homomorphism $\Delta\colon A \to A \otimes A$. Assume that A is generated, as a normed algebra, by the matrix elements u_{ij} of its non-degenerate finite-dimensional representations. Then (A, Δ) is a compact quantum group.*

Proof. Let $U = (u_{ij})$ be a finite dimensional representation of A. Then

$$\begin{aligned}(\Delta \otimes \mathrm{id})\Delta(u_{ij}) &= \sum_k \Delta(u_{ik}) \otimes u_{kj} \\ &= \sum_{k,l} u_{il} \otimes u_{lk} \otimes u_{kj} \\ &= \sum_l u_{il} \otimes \Delta(u_{lj}) \\ &= (\mathrm{id} \otimes \Delta)\Delta(u_{ij})\end{aligned}$$

for all generators, so Δ is coassociative.

Let $U = (u_{ij})$ be a non-degenerated finite dimensional representation of A, and denote by $W = (w_{ij})$ its inverse. Then by simple calculations we obtain

$$\sum_k \Delta(u_{ik})(1 \otimes w_{kj}) = \sum_{k,l} u_{il} \otimes u_{lk} w_{kj} = \sum_l u_{il} \otimes \delta_{lj} 1 = u_{ij} \otimes 1.$$

Hence $u_{ij} \otimes 1 \in \Delta(A)(1 \otimes A)$ for all generators u_{ij}. Now suppose

$$a \otimes 1 = \sum_k \Delta(a_k^{(1)})(1 \otimes a_k^{(2)}) \text{ and } b \otimes 1 = \sum_l \Delta(b_l^{(1)})(1 \otimes b_l^{(2)})$$

with $a_k^{(1)}, a_k^{(2)}, b_l^{(1)}, b_l^{(2)} \in A$. Then

$$\begin{aligned}(a \otimes 1)(b \otimes 1) = (ab \otimes 1) &= \sum_k \Delta(a_k^{(1)})(1 \otimes a_k^{(2)})(b \otimes 1) \\ &= \sum_{k,l} \Delta(a_k^{(1)} b_l^{(1)})(1 \otimes a_k^{(2)} b_k^{(2)}).\end{aligned}$$

Therefore if $a \otimes 1, b \otimes 1 \in \Delta(A)(1 \otimes A)$, then $ab \otimes 1 \in \Delta(A)(1 \otimes A)$. Therefore since $u_{ij} \otimes 1 \in \Delta(A)(1 \otimes A)$ we obtain

$$A \otimes 1 \subseteq \overline{\Delta(A)(1 \otimes A)} \subseteq A \otimes A.$$

Since we can write $x \otimes y = (x \otimes 1)(1 \otimes y)$, we obviously obtain $\overline{\Delta(A)(1 \otimes A)} = A \otimes A$. Similarly we get $\overline{\Delta(A)(A \otimes 1)} = A \otimes A$. □

Corollary 3.3.8 (CMQG are CQG): *Let $G = (C(G), u)$ be a compact matrix quantum group, then $(C(G), \Delta)$ is a compact quantum group.*

Proof. Follows from the definition of a compact matrix quantum group and Proposition 3.3.7. Note that by assumption u and $(u^T)^*$ are non-degenerate representations.□

Before proving a kind of algebraic picture of compact quantum groups, let us introduce the *right regular representation*, which we obtain by the Haar state, as done by Woronowizc in [Wor98]. For this denote by $(\mathcal{H}, \pi, \xi)$ the GNS-representation of h.

Definition 3.3.9 (Right regular representation): Let G be a compact quantum group and let $\mathcal{J}$ be a Hilbert space on which $C(G)$ acts faithfully and non-degeneratively.

The unitary representation $\underline{U} \in M(\mathcal{K}(\mathcal{H}) \otimes C(G))$ defined by

$$\underline{U}(\pi(a)\xi \otimes \eta) = \Delta(a)(\xi \otimes \nu)$$

for all $a \in C(G)$ and $\eta \in \mathcal{J}$, is called *right regular representation.*

As notation we also write $a\xi$ for $\pi(a)\xi$.

Note that $\underline{U}$ is by definition on $\mathcal{H} \otimes \mathcal{J}$ and it is not directly clear why such a representation exists, but one can show everything stated as in [Wor98; MV98].

Proposition 3.3.10: *Let G be a compact quantum group and denote by $\underline{U}$ its right regular representation. Then the following is true*

(i) *The regular representation $\underline{U}$ implements Δ in the following sense: For all $a \in C(G)$ we have*

$$\Delta(a) = \underline{U}(a \otimes 1)\underline{U}^*.$$

(ii) *The set*

$$\{(\omega \otimes 1)(\underline{U}) \mid \omega \in \mathcal{K}(\mathcal{H}, \mathcal{J})^*\} \subseteq A$$

is dense in A.

Proof. Let $\mathcal{J}$ be as in the definition of the right regular representation.

For the first part, let $a, b \in C(G)$ and $\eta \in \mathcal{J}$, then

$$\underline{U}(a \otimes 1)(b\xi \otimes \eta) = \underline{U}(ab\xi \otimes \eta) = \Delta(ab)(\xi \otimes \eta) = \Delta(a)\Delta(b)(\xi \otimes \eta) = \Delta(a)\underline{U}(b\xi \otimes \eta)$$

and in particular $\underline{U}(a \otimes 1) = \Delta(a)\underline{U}$, i.e. $\Delta(a) = \underline{U}(a \otimes 1)\underline{U}^*$.

For the proof of the second statement, see for instance [Wor98; MV98]. □

Both parts of the last proposition highlight the speciality of the right regular representation, since we can recover by this our compact quantum group.

Moreover we can prove that every irreducible representation is contained in the right regular representation. The proof will be done as in [MV98] by using Lemma 3.3.4.

Let $\omega_{\xi_1,\xi_2} \in \mathcal{B}(\mathcal{H})^*$ be a continuous linear functional defined by

$$\omega_{\xi_1,\xi_2}(x) = \langle x\xi_1, \xi_2\rangle$$

for $x \in \mathcal{B}(\mathcal{H})$ and $\xi_1, \xi_2 \in \mathcal{H}$.

Proposition 3.3.11: *Let G be a compact quantum group and V be any irreducible unitary representation of G acting on a Hilbert space $\mathcal{J}$. Then V is contained in $\underline{U}$.*

Proof. Let $x \in \mathcal{K}(\mathcal{H}, \mathcal{J})$, consider

$$y = (\mathrm{id} \otimes h)(V^*(x \otimes 1)\underline{U}).$$

By Lemma 3.3.4 we have $y \in \mathcal{K}(\mathcal{H}, \mathcal{J})$ and $(y \otimes 1)\underline{U} = V(y \otimes 1)$. Let p be the projection on the image of y, then

$$(p \otimes 1)V(y \otimes 1) = (p \otimes 1)(y \otimes 1)\underline{U} = (py \otimes 1)\underline{U} = (y \otimes 1)\underline{U} = V(y \otimes 1)$$

and thus since p is a projection on the range of y we get

$$(p \otimes 1)V(p \otimes 1) = V(p \otimes 1).$$

By assumption V was irreducible, therefore $p = 0$ or $p = 1$, and in particular $y = 0$ or y surjective.

If y is surjective then V is equivalent to a subrepresentation of $\underline{U}$, since y is in particular unitary and since $\mathcal{J}$ is finite-dimensional.

Now let $y = 0$. Assume that we take $x \in \mathcal{K}(\mathcal{H}, \mathcal{J})$ such that for all $\xi \in \mathcal{H}$ we have $x\xi = \langle \xi, \xi_1\rangle\eta_1$ for some $\xi_1 \in \mathcal{H}$ and $\eta_1 \in \mathcal{J}$. Then we get for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{J}$ that

$$(V^*(x \otimes 1)\underline{U})(\xi \otimes \eta) = (V^*(1 \otimes a))(\eta_1 \otimes \eta)$$

with $a = (\omega_{\xi,\xi_1} \otimes 1)u$. Thus $y = 0$ implies $(\mathrm{id} \otimes h)(V^*(1 \otimes a))(\eta_1) = 0$. Because this is true for any η_1 we get

$$(\mathrm{id} \otimes h)(V^*(1 \otimes a)) = 0$$

for all a of the form $a = (\omega_{\xi,\xi_1} \otimes \mathrm{id})\underline{U}$. In Proposition 3.3.10 we have seen that these elements are dense in $C(G)$. Therefore

$$(\mathrm{id} \otimes h)(V^*(1 \otimes a)) = 0$$

for all $a \in A$. If we multiply to the right with any $x \in \mathcal{B}(\mathcal{J})$ and use linearity we find also that $(\mathrm{id} \otimes h)(V^*V) = 0$, which leads us to a contradiction as $V^*V = 1$. $\square$

Now we want to introduce Hopf *-algebras, which will yield us a suitable algebraic picture.

Definition 3.3.12 (Hopf *-algebra): A *Hopf *-algebra* consists of a unital *-algebra A together with

(i) a comultiplication $\Delta\colon A \to A \otimes A$, which is a *-homomorphism,

(ii) a *-homomorphism $\varepsilon\colon A \to \mathbb{C}$, such that $(\varepsilon \otimes \mathrm{id})\delta = \mathrm{id} = (\mathrm{id} \otimes \varepsilon)\delta$, the so called *counit*,

(iii) a linear map $S\colon A \to A$, such that $\mu(S \otimes id)\delta = \mu(\mathrm{id} \otimes S)\delta = \eta \circ \varepsilon$, the so called *antipode*. Here $\mu\colon A \otimes A \to A$ denotes the multiplication $a \otimes b \mapsto ab$ and $\eta\colon \mathbb{C} \to A$ the natural embedding $\lambda \mapsto \lambda 1$.

We now can prove the algebraic picture of compact quantum groups.

Proposition 3.3.13 (Algebraic picture of compact quantum groups): *Let G be a compact quantum group. Let G_0 be the subspace of $C(G)$ spanned by the matrix elements of all finite dimensional unitary representations of G. Then*

(i) *$G_0 \subseteq C(G)$ is a dense *-algebra,*

(ii) *$\Delta(G_0) \subseteq G_0 \odot G_0$, where $\odot$ denotes the algebraic tensor product,*

(iii) *$(G_0, \Delta|_{G_0})$ is a Hopf *-algebra.*

Proof. We only want to prove the first and the second part of the statement. For (iii) see for instance [MV98].

(i) Note that the product of two matrix coefficients of finite dimensional unitary representations is contained in the tensor product of the corresponding representations. Moreover the adjoint of a finite dimensional unitary representation is equivalent to a unitary representation. Thus G_0 is a *-subalgebra of $C(G)$.

By Proposition 3.3.11, we know that the regular representation $\underline{U}$ decomposes in irreducible unitary representations. Denote $\{U_\alpha \mid \alpha \in I\}$ for the irreducible unitary representations and $\mathcal{H}_\alpha$ for the corresponding Hilbert spaces. For $\alpha \in I$, let $n(\alpha)$ be the dimension of $\mathcal{H}_\alpha$ and let $\{\xi_1^\alpha, \xi_2^\alpha, \dots, \xi_{n(\alpha)}^\alpha\}$ be an orthonormal basis for $\mathcal{H}_\alpha$. For $\alpha, \beta \in I$ and for $x \in K(\mathcal{H})$ define $\omega_{pq}^{\alpha\beta} \in K(\mathcal{H})^*$

$$\omega_{pq}^{\alpha\beta}(x) = \langle x\xi_p^\alpha, \xi_q^\beta \rangle.$$

The linear span of

$$\{\omega_{pq}^{\alpha\beta} \mid \alpha, \beta \in I,\ 1 \leq p \leq n(\alpha),\ 1 \leq q \leq n(\beta)\}$$

is dense in $K(\mathcal{H})^*$. By Proposition 3.3.10 we know that

$$\{(\omega_{pq}^{\alpha\beta} \otimes 1)(\underline{U}) \mid \alpha, \beta \in I,\ 1 \leq p \leq n(\alpha),\ 1 \leq q \leq n(\beta)\}$$

is dense in $C(G)$. Now note that $(\omega^{\alpha\alpha} \otimes 1)\underline{U}$ are the matrix elements of the representation of U_α, which is contained in U. For $\alpha \neq \beta$ we have that $(\omega^{\alpha\beta} \otimes 1)\underline{U} = 0$. Thus we obtain the statement.

(ii) It suffices to check the statement for monomials of the form

$$U^{\alpha_1}_{i_1 j_1} \dots U^{\alpha_n}_{i_n j_n} \in G_0,$$

where $n \in \mathbb{N}$. Then

$$\begin{aligned} \Delta(U^{\alpha_1}_{i_1 j_1} \dots U^{\alpha_n}_{i_n j_n}) &= \prod_{l=1}^{n} \Delta(U^{\alpha_l}_{i_l j_l}) \\ &= \sum_{k_1,\dots,k_n} \prod_{l=1}^{n} U^{\alpha_l}_{i_l k_l} \otimes \prod_{l=1}^{n} U^{\alpha_l}_{k_l j_l} \in G_0 \odot G_0. \end{aligned}$$

Thus monomials are mapped to $G_0 \odot G_0$, and by this the statement follows. □

We also write

$$\mathrm{Pol}(G) := G_0 = \{u^{\alpha}_{ij} \mid \alpha \in \mathrm{Irr}(G)\}$$

for the subspace of G spanned by the matrix elements of all finite-dimensional unitary representations.

We now want to define the *full C^*-algebra of functions on G*.

Definition 3.3.14 (Full C^*-algebra): Let G be a compact quantum group. The completion of $\mathrm{Pol}(G)$ under the *universal norm* such that

$$\|a\| := \sup\{\|\pi(a)\| \mid \pi \text{ is a representation of } \mathrm{Pol}(G)\}$$

for $a \in \mathrm{Pol}(G)$, is called *universal* or *full C^*-algebra of functions on G*, and is denoted by $C_f(G), C_u(G)$ or simply $C(G)$.

By using universality one can obtain a comultiplication Δ_u, such that $(C_u(G), \Delta_u)$ is also a compact quantum group. Moreover G is a compact quantum subgroup of $(C_u(G), \Delta_u)$.

Remark 3.3.15: We now have three different structures, namely $C_r(G), C(G)$ and $\mathrm{Pol}(G)$ to describe a compact quantum group. It turns out that a compact quantum groups can be equivalently described by all of them. Moreover one can show that every Hopf *-algebra (G_0, Δ) can be extended in some sense to a compact quantum group $(C(G), \Delta)$. Conversely we already saw that every compact quantum group $(C(G), \Delta)$ has a "canonical" dense Hopf *-algebra (G_0, Δ) sitting inside it.

Moreover in the sense of category theory we obtain a natural equivalence for the case of full and a bijective correspondence in the reduced case:

$$\{\text{Hopf }^*\text{-algebras } \mathrm{Pol}(G)\} \xleftrightarrow{\cong} \{\text{Full compact quantum groups}(C(G), \Delta)\}$$

$$\{\text{Hopf }^*\text{-algebras } \mathrm{Pol}(G)\} \xleftrightarrow{1:1} \{\text{Red. compact quantum groups}(C_r(G), \Delta_r)\}$$

Definition 3.3.16 (Co-amenable): Let G be a compact quantum group. If the canonical surjection $\lambda_G\colon C(G) \to C_r(G)$ is an isomorphism, then G is called *co-amenable.*

We also want to establish the notion of *dual quantum subgroup* via the next lemma which we need to define the free wreath product with amalgamation.

Lemma 3.3.17: *Let G and H be compact quantum groups. Then the following is equivalent*

(i) $\iota\colon C(H) \to C(G)$ *is a faithful unital* **-homomorphism, intertwining the comultiplications,*

(ii) $\iota\colon C_r(H) \to C_r(G)$ *is a faithful unital* **-homomorphism, intertwining the comultiplications,*

(iii) $\iota\colon \mathrm{Pol}(H) \to \mathrm{Pol}(G)$ *is a faithful unital* **-homomorphism, intertwining the comultiplications.*

Definition 3.3.18 (Dual quantum subgroup): Let G and H be compact quantum groups. If there exists a faithful *-homomorphism $\iota\colon C(H) \to C(G)$ intertwining the comultiplications (or equivalently for $C_r(H), \mathrm{Pol}(H)$ like in Lemma 3.3.17), then we view $C(H) \subseteq C(G)$, $C_r(H) \subseteq C_r(G)$ and $\mathrm{Pol}(H) \subseteq \mathrm{Pol}(G)$ and call H *dual quantum subgroup* of G.

4. The dual discrete quantum group

Let G be a (locally) compact abelian group. Define

$$\widehat{G} := \{\varphi\colon G \to \mathbb{T} \mid \varphi \text{ is a group homomorphism}\}$$

as the *Pontryagin dual.* Pontryagin [Pon34] proved that every locally compact abelian group is isomorphic to its bidual, i.e. $G \cong \widehat{\widehat{G}}$. Moreover one can show that every abelian group G is compact if and only if $\widehat{G}$ is discrete.

This notion of duality fails, when we look at non-abelian compact groups. Since compact quantum groups are in some sense a generalisation of compact groups, we can also find a generalisation of Pontryagin duality for compact quantum groups. It will also turn out that our dual quantum group is *discrete.*

However there are multiple ways to introduce duality for compact quantum groups. We want to briefly give an algebraic approach as done in [Tim08; FSS17] for the dual discrete quantum group. But we also want to construct it by using the right regular representation, which is nowadays not the classical way to do it, since the regular representation or the so called *multiplicative unitary* are difficult to handle. However it is useful to construct the dual.

Definition 3.4.1 (Algebraic/discrete quantum group): (i) An *algebraic quantum group* is a unital Hopf *-algebra with a positive left and right-invariant Haar state, defined as in Proposition 3.1.6.

(ii) A *discrete algebraic quantum group*, is an algebraic quantum group, such that its underlying *-algebra is isomorphic to an algebraic direct sum of matrix algebras $\bigoplus_i M_{N_i}(\mathbb{C})$.

To construct now the Pontryagin dual of an algebraic quantum group A with Haar state h, we set

$$\widehat{A} = \{h(\cdot\, a) \mid a \in A\} \subseteq A^*,$$

where A^* denotes the algebraic dual space. One then can equip $\widehat{A}$ with a suitable multiplication, comultiplication, counit and antipode, to get again an algebraic quantum group $\widehat{A}$, as it is done in [Tim08]. Moreover it is proven for an algebraic quantum group A, that $A \cong \widehat{\widehat{A}}$, and also we get that $\widehat{A}$ is a discrete algebraic quantum group.

We also can construct the dual, by using the right regular representation as follows.

Reminder 3.4.2: Every compact quantum group G has a right regular representation $\underline{U}$, Definition 3.3.9. This right regular representation contains all the information necessary to reconstruct the compact quantum itself, since for all $a \in C(G)$ we have

$$\Delta(a) = \underline{U}(a \otimes 1)\underline{U}^*$$

and the set

$$\{(\omega \otimes 1)(\underline{U}) \mid \omega \in \mathcal{K}(\mathcal{H}, \mathcal{J})^*\} \subseteq C(G)$$

is dense in $C(G)$, Proposition 3.3.10.

Now define $\widehat{G} = (C_0(\widehat{G}), \widehat{\Delta})$ via

$$C_0(\widehat{G}) := \overline{\{(1 \otimes \omega)(\underline{U}) \mid \omega \in \mathcal{K}(\mathcal{H}, \mathcal{J})^*\}},$$

and

$$\widehat{\Delta}\colon C_0(\widehat{G}) \to M(C_0(\widehat{G}) \otimes C_0(\widehat{G})),\ \widehat{\Delta}(a) = \sigma\underline{U}(a \otimes 1)\underline{U}^*\sigma.$$

Hereby denotes $M(C_0(\widehat{G}) \otimes C_0(\widehat{G}))$ the multiplier algebra, and σ the tensor product flip map.

One now can show that this object $\widehat{G}$ defines us the *dual discrete quantum group* for the compact quantum group G. The proof will be omitted.

5. The free wreath product of compact quantum groups

In this section we want to define the so *free wreath product* of a compact quantum group by the quantum symmetric group. It was defined by Bichon in [Bic04].

Reminder 3.5.1 (Free (amalgamated) product of C^*-algebras): Let A, B be unital C^*-algebras.

(i) The *free product* $A *_{\mathbb{C}} B$ of A and B is defined as the universal C^*-algebra generated by copies of A and B with no additional relations. More precisely the free product $A *_{\mathbb{C}} B$ is a unital C^*-algebra with unital embeddings $\iota_A : A \to A *_{\mathbb{C}} B$ and $\iota_B : B \to A *_{\mathbb{C}} B$ such that for each unital C^*-algebra C and *-homomorphisms $f : A \to C$ and $g : B \to C$, there exists a unique *-homomorphism $f *_{\mathbb{C}} g : A *_{\mathbb{C}} \to C$ such that the following diagram commutes

$$\begin{array}{ccccc} A & \xrightarrow{\iota_A} & A *_{\mathbb{C}} B & \xleftarrow{\iota_B} & B \\ & {\scriptstyle f} \searrow & \Big\downarrow {\scriptstyle f *_{\mathbb{C}} g} & \swarrow {\scriptstyle g} & \\ & & C & & \end{array} .$$

(ii) Let D be a C^*-subalgebra of A and B, and denote by ϕ_A respectively ϕ_B the embedding of D in A, respectively B. The *free amalgamated product* $A *_D B$ is the universal C^*-algebra generated by the copies of A and B with the copies of D identified, i.e. $\phi_A(d) = \phi_B(d)$ for all $d \in D$.

Using this definition we can also define us the free (amalgamated) product for compact quantum groups, as done in [Wan95]. Note that the main problem is to obtain a suitable comuplitication on the free product of C^*-algebras. We will restrict ourselves to simply taking a look at the case without amalgamation, but everything can be generalised without problems, as in [Wan95].

We denote $A * B := A *_{\mathbb{C}} B$ for two unital C^*-algebras A and B

Proposition 3.5.2: *Let G, H be compact quantum groups. Then there exists a unique comultiplication Δ on $C(G) * C(H)$ such that $(C(G) * C(H), \Delta)$ is a compact quantum group and ι_G and ι_H are compact quantum group homomorphisms. We denote this compact quantum group by $G * H := (C(G) * C(H), \Delta)$, and call it the free product of compact quantum groups.*

Proof. We want to construct the unique comultiplication via the universal property at the level of C^*-algebras. For this let $C = (C(G) * C(H)) \otimes (C(G) * C(H))$ and let $f : C(G) \to C$ and $g : C(H) \to C$ be defined by $f = (\iota_G \otimes \iota_G)\Delta_A$ and $g = (\iota_H \otimes \iota_H)\Delta_B$. Then by the universal property we obtain a map $\Delta := f * g$. By coassociativity of Δ_G and Δ_H it easily follows that Δ is coassociative.

The C^*-algebra $C(G) * C(H)$ is generated by $\iota_G(C(G))$ and $\iota_H(C(H))$. Moreover by Proposition 3.3.13 we know that G and H are generated by the matrix elements of its non-degenerate finite-dimensional (in particular unitary) representations. If U is a non-degenerate finite-dimensional unitary representation of G respectively H, then also $\tilde{\iota}_G(U) = (\iota_G(u_{ij}))$ or $\tilde{\iota}_H(U) = (\iota_H(u_{ij}))$ are non-degenerate finite-dimensional unitary representations of $G * H$. Then use Proposition 3.3.7 to obtain the statement. $\square$

In the case of compact matrix quantum groups, there is a simple description of the free product.

Corollary 3.5.3: *Let $G = (C(G), u)$ and $H = (C(H), v)$ be compact matrix quantum groups, then so is $G*H = (C(G)*C(H), \tilde{\iota}_G(u) \oplus \tilde{\iota}_H(v))$, where $\tilde{\iota}_G$ and $\tilde{\iota}_H$ are defined as in the proof of Proposition 3.5.2. Moreover the comultiplication is the same.*

We also want to give explicitly the Haar state of $G * H$ for two compact quantum groups G, H. To do this let us introduce the free product of states defined by Voiculescu in [VDN92].

Definition 3.5.4 (Free product of states): Let φ, ψ be states on C^*-algebras A, B. The unique state $\varphi * \psi$ on $A * B$ is called *free product of states* if

(i) $(\varphi * \psi)\iota_A = \varphi$ and $(\varphi * \psi)\iota_B = \psi$,

(ii) if $c_1, \ldots, c_n$ are elements of $\ker \varphi$ or $\ker \psi$ and no adjecent elements belong to the same C^*-algebra A or B, then $c_1 \ldots c_n \in \ker(\varphi * \psi)$.

The Haar state is now given by the free product of the Haar states, see for instance [Tim08, Prop. 6.3.3].

Proposition 3.5.5: *Let G, H be compact quantum groups with Haar states h_G, and h_H, respectively. Then the free product of states $h_G * h_H$ is the Haar state of $G * H$.*

Also one can easily determine what the irreducible representations on $G * H$ are. Indeed take a family of irreducible unitary representations $(U^\alpha)_{\alpha \in \mathrm{Irr}(G)}$ from G and $(V^\beta)_{\alpha \in \mathrm{Irr}(H)}$ from H, then a family of representation on $G * H$ is given by simply taking the tensor products elements that belong to $(\tilde{\iota}_G(U^\alpha))_\alpha$ or $(\tilde{\iota}_H(V^\beta))_\beta$.

By this and the formula for the Haar state on $G * H$ we can also deduce the following statement, as done in [Tim08].

Corollary 3.5.6: *Let G and H be compact quantum groups. Then the following hold*

(i) $(C(G) * C(H), \Delta_G * \Delta_H)_u \cong (C_u(G), \Delta_G) * (C_u(H), \Delta_H)$,

(ii) $(C(G) * C(H), \Delta_G * \Delta_H)_r \cong (C_r(G), \Delta_G) * (C_r(H), \Delta_H)$, *where $(C(G) * C(H), \Delta_G * \Delta_H)_r$ denotes the reduced free product. Note that the underlying C^*-algebra is constructed as the image under the GNS-representation for $h_G * h_H$.*

To finish the theory on free products, we should briefly introduce amalgamated free products. as already mentioned, all statements work for this as well. To do so, we use a corollary of Wang [Wan95].

Definition 3.5.7 (Amalgamated free products): Let G, H be compact quantum groups, and D be a compact quantum subgroup of G and H, with natural embeddings j_G and j_H. Denote by $\langle D\rangle$ the closed two sided-ideal of

$$\iota_A j_A(d) - \iota_B j_B(d)$$

for $d \in C(D)$. Then the *amalgamated free product* of G and H under D is defined as

$$G *_D H := G * H/\langle D\rangle.$$

Now after introducing the basics of free products of compact quantum groups, we want to have a look at the *free wreath product* introduced by Bichon. We denote $C(G)^{*N} := C(G) * \ldots * C(G)$ (N-times) for a C^*-algebra $C(G)$. Similarly for the amalgamated free product.

Definition 3.5.8 (Free wreath product): Let G be a compact quantum group, and $n \in \mathbb{N}$. The *free wreath product* of G by S_N^+ is defined by

$$C(G \wr_* S_N^+) = C(G)^{*N} * C(S_N^+)/I,$$

where we consider the full free product and I is the two-sided closed ideal generated by

$$\nu_k(a)u_{ki} - u_{ki}\nu_k(a),\ 1 \leq k, i \leq n,\ a \in C(G),$$

where u_{ki} are the matrix coefficients of the fundamental representation of S_N^+ and we denote $\nu_k \colon C(G) \to C(G)^{*N} \subseteq C(G)^{*N} * C(S_N^+)$ for the canonical *-homomorphism embedding in the k-th copy.

Remark 3.5.9: If G is a quantum permutation group with fundamental representation v, then $G \wr_* S_N^+$ is also a quantum permutation group with magic unitary defined by $w_{pi,qj} = \nu_i(v_{pq})u_{ij}$.

Notice, that the notation here might be slightly confusing, since we use tensor identifications, and note that we might need to embed the free wreath product in S_M^+ with $M \geq N$, to get that $G \wr_* S_N^+$ is a quantum permutation group.

Following Freslon [Fre23], we can also define a free wreath product with amalgamation, in the following sense.

Definition 3.5.10: Let G be a compact quantum group, and H be a dual quantum subgroup of G. The *free amalgamated wreath product* of G is defined as

$$C(G \wr_{*,H} S_N^+) = C(G)^{*_H N} * C(S_N^+)/I,$$

where I is the same two-sided closed ideal as in Definition 3.5.8.

With a suitable comultiplication the free wreath product of a compact quantum group G is also a compact quantum group. See [Bic04].

Proposition 3.5.11 (Free wreath products are CQG): *Let G be a compact quantum group, and $N \in \mathbb{N}$, then $G \wr_* S_N^+$ is a compact quantum group with the comultiplication Δ satisfying the following*

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj},\ \Delta(\nu_i(a)) = \sum_{k=1}^{n} \nu_i \otimes \nu_k(\Delta(a))(u_{ik} \otimes 1)$$

We now want to give a proof why we can indeed write $H_N^+ \cong \widehat{\mathbb{Z}/2\mathbb{Z}} \wr_* S_N^+$. This proof can be generalised for the *quantum reflection group* H_N^{s+} for $s \geq 1$.

For this let us introduce *sudokus*, as it is done in [BV09].

Definition 3.5.12 (Sudoku): Let $s, n \in \mathbb{N}$. A (s,n)-*sudoku* is a magic unitary of size sn of the form

$$m = \begin{pmatrix} a^0 & a^1 & \dots & a^{s-1} \\ a^{s-1} & a^0 & \dots & a^{s-2} \\ \vdots & \vdots & \ddots & \vdots \\ a^1 & a^2 & \dots & a^0 \end{pmatrix},$$

where $a^0, \dots, a^{s-1} \in M_n(\mathbb{C})$. Note that m is *circulant* and we can write $m = (a_{ij}^{p-q})_{pi,qj}$ under using tensor product identifications, where all indices are modulo s.

Banica and Vergnioux proved in [BV09, Theorem 2.3] that

$$C(H_N^+) \cong C^*(a_{ij}^p \mid (a_{ij}^{p-q})_{pi,qj} \text{ is } (2,N)\text{-sudoku}).$$

Moreover H_N^+ is a quantum permutation group of S_{2N}^+.

Lemma 3.5.13: *We have the following identification*

$$\widehat{\mathbb{Z}/2\mathbb{Z}} \cong C(\mathbb{Z}/2\mathbb{Z}) \cong C^*(\mathbb{Z}/2\mathbb{Z}) = C^*\left(p, q \middle| \begin{pmatrix} p & q \\ q & p \end{pmatrix} \textit{ is a magic unitary}\right).$$

Moreover $C^*(\mathbb{Z}/2\mathbb{Z})$ is a quantum permutation group.

Proof. The first isomorphism is clear, since $C^*(\mathbb{Z}/2\mathbb{Z})$ is commutative and of dimension 2, therefore $C^*(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{C} \oplus \mathbb{C}$, and $C(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{C} \oplus \mathbb{C}$ as well. For the second isomorphism, note that

$$C^*(\mathbb{Z}/2\mathbb{Z}) = C^*(1, u_1 \mid u_1^2 = 1) = C^*(1, p, q \mid p + q = 1),$$

where in the last step p, q are projections defined by $p = \frac{1}{2}(1 + u_1)$ and $q = \frac{1}{2}(1 - u_1)$. Thus we can also write

$$C^*(\mathbb{Z}/2\mathbb{Z}) = C^*\left(p, q \middle| \begin{pmatrix} p & q \\ q & p \end{pmatrix} \text{ is a magic unitary}\right).$$

Hence $C^*(\mathbb{Z}/2\mathbb{Z})$ is a quantum permutation group. ☐

Using these facts we want to prove that the quantum hyperoctahedral group can be seen as a free wreath product.

Proposition 3.5.14: *Let $N \in \mathbb{N}$, then we have the following isomorphism*

$$C(H_N^+) \cong C(\mathbb{Z}/2\mathbb{Z}) \wr_* C(S_N^+).$$

Proof. Firstly note that $\mathbb{Z}/2\mathbb{Z}$ can be written as $\mathbb{Z}/2\mathbb{Z} \cong <\sigma> \subseteq \mathrm{GL}_2(\mathbb{C})$, where

$$\sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is cyclic with $\sigma^2 = I_2$. Taking the isomorphism $C^*(\mathbb{Z}/2\mathbb{Z}) \cong C(\mathbb{Z}/2\mathbb{Z})$ and by the lemma before $C(\mathbb{Z}/2\mathbb{Z})$ is a quantum permutation group, where our fundamental representation $v = (v_{pq})_{0 \leq p,q \leq 1}$ is defined by

$$v_{pq} = \begin{cases} \sigma & \mapsto p+q \mod 2 \\ \sigma^2 & \mapsto p+q+1 \mod 2 \end{cases}.$$

Observe that v is a circulant matrix as one can easily verify, i.e. $v_{pq} = v_{0(p-q)}$.

One then defines a map $\Phi\colon C(H_N^+) \to C(\mathbb{Z}/2\mathbb{Z}) \wr_* C(S_N^+)$. Considering the generators of the magic unitary as in Remark 3.5.9 and using the definition of $C(H_N^+)$ as universal C^*-algebra, we claim that the elements of the following form are forming a sudoku:

$$A_{ij}^p = \nu_i(v_{0p})u_{ij}.$$

Indeed the corresponding matrix defined in Definition 3.5.12, is given by

$$m_{pi,qj} = A_{ij}^{p-q} = \nu_i(v_{0(p-q)}u_{ij} = \nu_i(v_{pq})u_{ij} = w_{pi,qj},$$

where we used that v is a circulant matrix, and since the fundamental representation of the free wreath product is a magic unitary, we proved the claim, and obtain a *-homomorphism sending generators to generators.

We now want to construct an inverse *-homomorphism $\Psi\colon C(\mathbb{Z}/2\mathbb{Z}) \wr_* C(S_N^+) \to C(H_N^+)$. To do this, we define suitable *-homomorphisms from $C(\mathbb{Z}/2\mathbb{Z}) \to C(H_N^+)$ respectively $C(S_N^+) \to C(H_N^+)$ and prove that the generators of this *-homomorphisms satisfy the defining relation of the free wreath product. Let a_{ij}^p be the generators of $C(H_N^+)$ then define elements $U = (U_{pq})$ and $V = (V_{ij})$ by

$$U_{pq} = \sum_k a_{ik}^{p-q},$$
$$V_{ij} = \sum_r a_{p-q}^r.$$

Then U defines us a *-homomorphism $C(\mathbb{Z}/2\mathbb{Z}) \to C(H_N^+)$ for every copy of $C(\mathbb{Z}/2\mathbb{Z})$ and V defines us a *-homomorphism $C(S_N^+) \to C(H_N^+)$, where we used the fact that $C(H_N^+)$ is a quantum permutation group, therefore this map makes sense without further justification.

Then one can compute $\nu_i(U_{pq})V_{ij} - V_{ij}\nu_i(U_{pq}) = 0$, such that we obtain a *-homomorphism Ψ via $\nu_i(U)$ and V.

One then can check as in [BV09] that Ψ is indeed the inverse of Φ. This proves the statement. □

Chapter IV.

Graphs of C^*-algebras

We follow the definition of a graph in the sense of Serre [Ser77]. Let $\mathcal{G}$ be a graph. We denote by $V(\mathcal{G})$ the *vertex set* of $\mathcal{G}$, and its *edge set* by $E(\mathcal{G})$. For $e \in E(\mathcal{G})$ we denote by $s(e)$ and $r(e)$ respectively its *source* and *range* of e and by $\overline{e}$ its *inverse edge*. Note that we use geometric edges, i.e. for all $e \in E(\mathcal{G})$ we also have $\overline{e} \in E(\mathcal{G})$. We say that $E(\mathcal{G}) = E^+(\mathcal{G}) \cup E^-(\mathcal{G})$, with $e \in E^+(\mathcal{G})$ if and only if $\overline{e} \in E^-(\mathcal{G})$, is a *partition*. Moreover we call $\mathcal{H} \subseteq \mathcal{G}$ a *connected subgraph* if $V(\mathcal{H}) \subseteq V(\mathcal{G})$, $E(\mathcal{H}) \subseteq E(\mathcal{G})$ such that $e \in E(\mathcal{H})$ if and only if $\overline{e} \in E(\mathcal{H})$ and the source and range map from $\mathcal{G}$ are restricted to $\mathcal{H}$.

For the whole chapter $\mathcal{G}$ is a graph.

1. Graphs of C^*-algebras and the maximal fundamental C^*-algebra

Graphs of groups are a basic tool used in Bass-Serre theory for groups. In the following motivation we want to recall basic ideas of it. Our goal is to establish some kind of "quantum Bass-Serre theory", which will help us computing the K-theory, by using the so called fundamental C^*-algebra.

In [Ver04], Vergnioux first used quantum Bass-Serre trees in spirit of [JV84] to obtain results about the K-theory and especially K-amenability of free amalgamated products. This was then extended by Fima and Freslon [FF14], who proved that the fundamental group of discrete quantum groups are K-amenable. Nevertheless we will restrict ourselves to graph of C^*-algebras.

Motivation 4.1.1: In [Ser77] introduced *graph of groups*, this is a graph where each vertex and edge is assigned a group. The groups assigned to edges are related to the groups assigned to the vertices they connect with via injective group homomorphisms. Serre then defined the so called *fundamental group of a graph of groups*, which is a generalisation of the topological fundamental group, and captures the structure of the graph of groups in some algebraic sense.

The fundamental theorem of Bass-Serre theory says that every group acting on a tree without inversion, is isomorphic to a fundamental group of a graph of groups, namely the quotient graph of groups.

In analogy to the definition by Serre we now define graph of C^*-algebras.

Definition 4.1.2 (Graph of C^*-algebras): A *graph of C^*-algebras* is a tuple

$$(\mathcal{G}, (A_q)_{q\in V(\mathcal{G})}, (B_e)_{e\in E(\mathcal{G})}, (s_e)_{e\in E(\mathcal{G})})$$

such that

(i) $\mathcal{G}$ is a connected graph,

(ii) for all $q \in V(\mathcal{G})$ and $e \in E(\mathcal{G})$, A_q and B_e are unital C^*-algebras,

(iii) for all $e \in E(\mathcal{G})$ we have $B_{\overline{e}} = B_e$ and

(iv) for all $e \in E(\mathcal{G})$, $s_e\colon B_e \to A_{s_e}$ is a unital faithful *-homomorphism.

For every $e \in E(\mathcal{G})$ we denote $r_e = s_{\overline{e}}\colon B_e \to A_{r(e)}$, $B_e^s = s_e(B_e), B_e^r = r_e(B_e)$.

For simplicity we write $(\mathcal{G}, (A_q)_{q\in V(\mathcal{G})}, (B_e)_{e\in E(\mathcal{G})})$ for a graph of C^*-algebras.

Remark 4.1.3 (Maximal subtree): Let $\mathcal{G}$ be a connected graph. One can show that every connected graph contains a subtree $\mathcal{T}_0 \subseteq \mathcal{G}$. By defining a suitable ordering, one can apply Zorn's Lemma and obtain a *maximal subtree* $\mathcal{T}$ of $\mathcal{G}$.

We now want to define the *maximal fundamental C^*-algebra* as an analogue of the fundamental group one get in the sense of Bass-Serre-theory.

Definition 4.1.4 (Fundamental C^*-algebra): Let $(\mathcal{G}, (A_q)_{q\in V(\mathcal{G})}, (B_e)_{e\in E(\mathcal{G})})$ be a graph of C^*-algebras and let $\mathcal{T}$ be a maximal subtree of $\mathcal{G}$. The *(maximal) fundamental C^*-algebra* with respect to $\mathcal{T}$ is the universal C^*-algebra generated by the C^*-algebras A_q for $q \in V(\mathcal{G})$ and by unitaries u_e for $e \in E(\mathcal{G})$ with the following relations

(i) for all $e \in E(\mathcal{G})$ we have $u_{\overline{e}} = u_e^*$,

(ii) for all $e \in E(\mathcal{G})$ and $b \in B_e$ we have $u_{\overline{e}} s_e(b) u_e = r_e(b)$ and

(iii) for all $e \in E(\mathcal{T})$ we have $u_e = 1$.

This C^*-algebra will be denoted as $\pi_1^{\max}(\mathcal{G}, (A_q)_{q\in V(\mathcal{G})}, (B_e)_{e\in E(\mathcal{G})}, \mathcal{T})$.

Since we are dealing with universal C^*-algebras, it is necessary to prove that the maximal fundamental C^*-algebra exists and is non-trivial. See for instance [FG18] for a proof.

Proposition 4.1.5: *Let $(\mathcal{G}, (A_q)_{q\in V(\mathcal{G})}, (B_e)_{e\in E(\mathcal{G})})$ be a graph of C^*-algebras and let $\mathcal{T}$ be a maximal subtree of $\mathcal{G}$, then its maximal fundamental C^*-algebra P is non-trivial. Moreover the canonical *-homomorphisms $A_q \to P$ are faithful for all $q \in V(\mathcal{G})$.*

For our next example we need the HNN-extension of C^*-algebras, [Ued05; Ued08], therefore we want to introduce it here. Let us quickly recall the construction for groups:

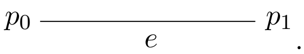
For groups the HNN-construction of a group G is a group Γ in which G embeds in such a way that two given isomorphic subgroups of G are conjugate. More precisely, given a subgroup $H \subseteq G$ and an injective group homomorphism $\theta\colon H \to G$, the HNN-extension is defined as

$$\Gamma = \langle G, t \mid t\sigma t^{-1} = \theta(\sigma) \text{ for all } \sigma \in H\rangle.$$

Definition 4.1.6 (Full HNN-extension): Let A be a unital C^*-algebra and $B \subseteq A$ a unital C^*-subalgebra. Moreover let $\theta\colon B \to A$ be an injective *-homomorphism.

The *full or universal* HNN-*extension* is the universal C^*-algebra generated by A and a unitary $u(\theta)$ such that $u(\theta)\theta(b)u(\theta)^* = b$ for all $b \in B$, denoting it by $\mathrm{HNN}(A, B, \theta)$.

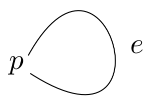
Example 4.1.7: (i) Let $\mathcal{G}$ be the following graph



Let $A_0 = A_{p_0}$, $A_1 = A_{p_1}$ and $B = B_e \subseteq A_0, A_1$ be unital C^*-algebras. Moreover let s_e respectively $s_{\overline{e}}$ be the canonical embedding of B into A_0 respectively A_1.

Obviously the maximal subtree of $\mathcal{G}$ is $\mathcal{G}$ itself, by this $u_e = 1$, and therefore only (ii) in Definition 4.1.4 provides us with a relation. Thus we conclude $\pi_1^{\max}(\mathcal{G}, (A_0, A_1), B, \mathcal{G}) \cong A_0 *_B A_1$.

(ii) Let $\mathcal{G}$ be the following graph



Let $A = A_p$ and $B = B_e$ be unital C^*-algebras and $\theta\colon B \to A$ be an injective *-homomorphism. We set $s_e = \mathrm{id}$ and $r_e = \theta$. Obviously the maximal subtree is $\{p\}$. By this we obtain the full HNN-extension $\mathrm{HNN}(A, B, \theta)$ as fundamental C^*-algebra of this graph of C^*-algebras.

2. Reduced fundamental C^*-algebras

We want to define the reduced (also sometimes called edge-reduced) fundamental C^*-algebra and the vertex-reduced C^*-algebra. To do so we equip the edges of a graph of C^*-algebras with conditional expectations $E_e^s\colon A_{s(e)} \to B_e^s := s_e(B_e)$ for all $e \in E(\mathcal{G})$.

The reason for having different notions of reduced fundamental C^*-algebras is the fact that the conditional expectations might behave "degenerate". If we assume that the conditional expectations are GNS-faithful, we obtain the (edge)-reduced fundamental C^*-algebra, otherwise if they are not necessarily GNS-faithful, we are in the setting of vertex-reduced fundamental C^*-algebras. Both were introduced in [FF14] and [FG18].

To define the reduced fundamental C^*-algebra, we need some kind of GNS-construction which we obtain from a conditional expectation, see for instance [Pas73]. For this recall the definition of a Hilbert C^*-module, Definition 2.1.7 and recall that conditional expectations are unital completely positive maps.

Construction 4.2.1 (GNS-construction): Let A and B be a unital C^*-algebras and $\phi\colon A \to B$ a unital completely positive map. There exists a triple $(\mathcal{H}, \rho, \xi)$, where $\mathcal{H}$ is a Hilbert B-module, $\xi \in \mathcal{H}$ and $\rho\colon A \to \mathcal{B}_B(\mathcal{H})$ is a unital *-homomorphism such that $\rho(A)\xi B$ is dense in $\mathcal{H}$ and $\phi(a) = \langle \xi, \rho(a)\xi \rangle$ for all $a \in A$. This construction is unique up to isomorphism.

If we take a unital C^*-subalgebra $B \subseteq A$ and a conditional expectation $E\colon A \to B$, we have that the Hilbert B-submodule ηB of the GNS-construction with respect to E is complemented in $\mathcal{H}$. Indeed we have

$$\mathcal{H} = \xi\mathcal{H} \oplus \mathcal{H}^\circ,$$

where

$$\mathcal{H}^\circ = \overline{\{\rho(a)\xi b \mid a \in \ker(E), b \in B\}}.$$

We say that E is *GNS-faithful* or *non-degenerate* if ρ is faithful.

Definition 4.2.2 (GNS-faithful): Let A be a unital C^*-algebra and $(B_i)_{i \in I}$ be a family of C^*-algebras. A family $(\varphi_i)_{i \in I}$ of unital completely positive maps $\varphi_i\colon A \to B_i$ is called *GNS-faithful* if $\bigcap_{i \in I} \ker(\pi_i) = \{0\}$, where (H_i, π_i, ξ_i) is the GNS-construction of φ_i.

We will use a slightly different notation for the GNS-construction, as we will set as third argument the natural linear map $\eta\colon A \to \mathcal{H}$, which sends 1 to ξ. We will then say that $(\mathcal{H}, \rho, \eta)$ is the GNS-construction.

Let $(\mathcal{G}, (A_q), (B_e))$ be a graph of C^*-algebras with GNS-faithful conditional expectations $E_e^s\colon A_{s(e)} \to B_e^s := s_e(B_e)$ for every edge $e \in E(\mathcal{G})$. Then denote for every edge $e \in E(\mathcal{G})$ by $(\mathcal{H}_e^s, \pi_e^s, \eta_e^s)$ the GNS-construction with respect to the completely positive map $s_e^{-1} \circ E_e^s$. Hence $\mathcal{H}_e^s$ is a Hilbert B_e-module, obtained as completion of $A_{s(e)}$ with respect to the induced B_e-valued inner product. The representation π_e^s is induced by left multiplication, moreover we define $\xi_e^s := \eta_e^s(1)$.

Since by assumption the conditional expectations E_e^s are GNS-faithful, we may identify $A_{s(e)}$ with its image in $\mathcal{B}_{B_e}(\mathcal{H}_e^s)$ via the representation π_e^s for all edges $e \in E(\mathcal{G})$. We now can construct *path Hilbert-modules*, it will carry the faithful representation of the reduced fundamental C^*-algebra.

Construction 4.2.3 (Path Hilbert C^*-modules): Let $n \in \mathbb{N}_0$ and let $w = (e_1, \ldots, e_n)$ be a path on $\mathcal{G}$. We define Hilbert C^*-modules

$$\mathcal{J}_i := \begin{cases} \mathcal{H}_{e_1}^s, & \text{if } i = 0 \\ \mathcal{H}_{e_{i+1}}^s, & \text{if } e_{i+1} \neq \overline{e_i} \\ \left(\mathcal{H}_{e_{i+1}}^s\right)^\circ, & \text{if } e_{i+1} = \overline{e_i} \\ A_{r(e_n)}, & \text{if } i = n. \end{cases}$$

We can construct a tensor product for these Hilbert C^*-modules, while using actions induced by the representations, for more details see [FF14, Section 3.2]. We then define

$$\mathcal{H}_w := \mathcal{J}_0 \otimes \ldots \otimes \mathcal{J}_n,$$

which we will call *path Hilbert module*, $\mathcal{H}_w$ is a Hilbert $A_{r(e_n)}$-module.

For two vertices $p, p_0 \in V(\mathcal{G})$ we set

$$\mathcal{H}_{p_0,p} = \bigoplus_{\text{Path } w \text{ from } p_0 \text{ to } p} \mathcal{H}_w,$$

which is a Hilbert A_p-module.

We need to construct suitable unitaries. For this let $e \in E(\mathcal{G})$ and $p \in V(\mathcal{G})$, then define

$$u_e^p \colon \mathcal{H}_{r(e),p} \to \mathcal{H}_{s(e),p},$$

by the following case distinction:

Let w be a path of length n from $r(e)$ to p and let $\zeta \in \mathcal{H}_w$. We have to look at the cases $n = 0$, $n = 1$ and $n \geq 2$.

- For $n = 0$, we have the empty path w, then set $u_e^p(\zeta) = \xi_e^s \otimes \zeta \in \mathcal{H}_w$,
- For $n = 1$, then $w = (e_1)$ and $\zeta = a \otimes \zeta'$ where $a \in A_{s(e_1)}$ (where we identify $A_{s(e)}$ with its image under the GNS-representation), and $\zeta' \in A_p$. If $e_1 \neq \overline{e}$, then put $u_e^p(\zeta) = \xi_e^s \otimes \zeta \in \mathcal{H}_{(e,e_1)}$, else define

$$u_e^p = \begin{cases} \xi_e^s \otimes \zeta \in \mathcal{H}_{(e,e_1)}, & \text{if } a \in (\mathcal{H}_{e_1}^s)^\circ \\ r_{e_1} \circ s_{e_1}^{-1}(a)\zeta' \in A_p, & \text{if } a \in B_{e_1}^s. \end{cases}$$

- For $n \geq 2$, then $w = (e_1, \dots, e_n)$ and $\zeta = a \otimes \zeta'$ with $a \in A_{s(e_1)}$ and $\zeta' \in \mathcal{J}_1 \dots \otimes \mathcal{J}_n$. If $e_1 \neq \overline{e}$, then $u_e^p(\zeta) = \xi_e^s \otimes \zeta \in \mathcal{H}_{(e,w)}$, else

$$u_e^p = \begin{cases} \xi_e^s \otimes \zeta \in \mathcal{H}_{(e,w)}, & \text{if } a \in (\mathcal{H}_{e_1}^s)^\circ \\ r_{e_1} \circ s_{e_1}^{-1}(a)\zeta' \in \mathcal{H}_{(e_2,\dots,e_n)}, & \text{if } a \in B_{e_1}^s. \end{cases}$$

As explained in [FF14], we can extend u_e^p to unitaries such that $(u_e^p)^* = u_{\overline{e}}^p$. Moreover we have an analogue of the property (ii) in the definition of the full fundamental C^*-algebra Definition 4.1.4, as for all edges $e \in E(\mathcal{G})$ and $b \in B_e$

$$u_{\overline{e}}^p s_e(b) u_e^p = r_e(b).$$

Lastly for a path $w = (e_1, \dots, e_n)$ and $p \in V(\mathcal{G})$, we denote

$$u_w^p = u_{e_1}^p u_{e_2}^p \dots u_{e_n}^p.$$

These unitaries are the suitable ones, we now can use to define the reduced fundamental C^*-algebra.

Definition 4.2.4 (Reduced fundamental C^*-algebra): Let $(\mathcal{G}, (A_q)_q, (B_e)_e)$ be a graph of C^*-algebras and let $p_0, p \in V(\mathcal{G})$. The *reduced fundamental C^*-algebra* rooted in p_0 in base p is the C^*-algebra

$$\pi_1^p(\mathcal{G}, p_0) := C^*((u_z^p)^* A_q u_w^p \mid q \in V(\mathcal{G}),\ w,\ z \text{ paths from } q \text{ to } p_0) \subseteq \mathcal{B}_{A_p}(\mathcal{H}_{p_0,p})$$

Our definition depends on two arbitrary vertices, this bring us a lot of different representations of the reduced fundamental C^*-algebra. However it can be proven, that the definition does not really depend on the vertex p_0, since we can construct an isomorphism if we have two different rooting in our definition by the connectedness of $\mathcal{G}$. Hence we will simply write $P(p_0)$ for the reduced fundamental C^*-algebra $\pi_1^{p_0}(\mathcal{G}, p_0)$.

If we now assume that the conditional expectations are not necessarily GNS-faithful, we obtain the notion of vertex-reduced fundamental C^*-algebras. Instead of defining the vertex-reduced fundamental C^*-algebra as done in [FF14], we want to use the description as a quotient, in order to save us the technical details.

Definition 4.2.5 (Reduced Operator): Let $(\mathcal{G}, (A_q)_{q \in V(\mathcal{G})}, (B_e)_{e \in E(\mathcal{G})})$ be a graph of C^*-algebras. An element a in the fundamental C^*-algebra P is called *reduced operator* from p to p for a $p \in V(\mathcal{G})$, if

$$a = a_0 u_{e_1} a_1 u_{e_2} \dots u_{e_n} a_n$$

where $n \geq 1$ and $(e_1, \dots, e_n)$ is path from p to p itself, $a_0 \in A_p$, $a_k \in A_{r(e_k)}$ and if $e_{k+1} = \overline{e}_k$ implies $E_{e_{k+1}}^s(a_k) = 0$.

Definition 4.2.6 (Vertex-reduced fundamental C^*-algebra): Let $(\mathcal{G}, (A_q)_{q\in V(\mathcal{G})}, (B_e)_{e\in E(\mathcal{G})})$ be a graph of C^*-algebras and denote by P its full fundamental C^*-algebra. The unital C^*-algebra P_r called the *vertex-reduced fundamental C^*-algebra*, is the quotient $\lambda\colon P \to P_r$ of P such that

(i) There exists a GNS-faithful family $\{E_p \mid p \in V(\mathcal{G})\}$ of unital completely positive maps $E_p\colon P_r \to A_p$ such that $E_p(\lambda(a)) = a$ for all $a \in A_p$ and $E_p(\lambda(b)) = 0$ for all reduced operators $b \in P$ from p to p,

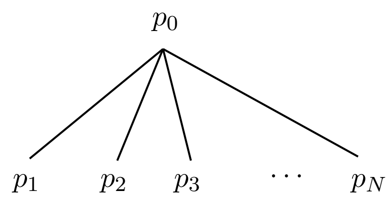
(ii) For any unital C^*-algebra C with a surjective *-homomorphism $\rho\colon P \to C$ and a GNS-faithful family $\{\varphi_p \mid p \in V(\mathcal{G})\}$ of unital completely positive maps $\varphi_p\colon C \to A_p$ such that $\varphi_p(\rho(a)) = a$ for all $a \in A_p$ and $\varphi_p(\rho(b)) = b$ for all reduced operators $b \in P$ from p to p, there exists a unique unital *-homomorphism $\nu\colon P_r \to C$ factorising ρ, i.e. $\nu \circ \lambda = \rho$.

3. Free wreath products as fundamental C^*-algebras

In this section we want to construct the full and vertex-reduced free wreath product as fundamental C^*-algebras. To do so we will mainly follow [FT24].

We denote $C_\bullet(\cdot)$ for either the full or the reduced C^*-algebra.

Let $N \in \mathbb{N}$, and $\mathcal{T}_N$ be a rooted tree with $N+1$ vertices $p_0, \dots, p_N$, where p_0 is the root, and $2N$ edges $v_1, \dots, v_N, \overline{v_1}, \dots, \overline{v_N}$ and source maps $s(v_k) = p_0$ and range maps $r(v_k) = p_k$ for all $1 \le k \le N$.



Equip the rooted tree $\mathcal{T}_N$ with the following setting

$$\begin{aligned} A_{p_0} &= C_\bullet(H) \otimes C_\bullet(S_N^+), \\ A_{p_k} &= C_\bullet(G) \otimes \mathbb{C}^N \text{ for } 1 \le k \le N, \\ B_{v_k} &= B_{\overline{v_k}} = C_\bullet(H) \otimes \mathbb{C}^N \text{ for } 1 \le k \le N, \end{aligned}$$

with source maps

$$s_{v_k}\colon C_\bullet(H) \otimes \mathbb{C}^N \to C_\bullet(H) \otimes L_k \subseteq C_\bullet(H) \otimes C_\bullet(S_N^+),\ h \otimes e_j \mapsto h \otimes u_{kj},$$

where $L_k := \operatorname{span}(u_{kj} \mid 1 \le j \le N)$ for $1 \le k \le N$. The range maps $r_{v_k}\colon C_\bullet(H) \otimes \mathbb{C}^N \to C_\bullet(H) \otimes \mathbb{C}^N$ are the canonical inclusions.

Following [FT24, Proposition 2.5. and 2.18.] we construct conditional expectations in the following way.

Proposition 4.3.1: *Let $1 \leq i \leq N$. The map*

$$E_i \colon C_\bullet(S_N^+) \to L_i,\ x \mapsto N \sum_{j=1}^{N} h(xu_{ij})u_{ij}$$

is a conditional expectation. Here h denotes the Haarstate of $C_\bullet(S_N^+)$.

Proof. First note that $L_i \subseteq \mathrm{Pol}(S_N^+)$ is a unital *-subalgebra of $\mathrm{Pol}(S_N^+)$. Moreover since $\mathbb{C}^N \to L_i, e_j \mapsto u_{ij}$ is a *-isomorphism, we may view L_i as a finite dimensional commutative C^*-subalgebra of $C_\bullet(S_N^+)$. One now can easily check linearity, since h is linear. Also positivity is clear, since u_{ij} are projections and h is a state. Moreover since $h(u_{ij}) = \frac{1}{N}$ we obtain

$$E_i(1) = N \sum_{j=1}^{N} h(u_{ij})u_{ij} = \sum_{j=1}^{N} u_{ij} = 1.$$

Since $\sum_{j=1}^{N} u_{ij} = 1$ is a partition of unity, we can conclude that two projections u_{il}, u_{ik} with $l \neq k$ are orthogonal. Therefore for arbitrary $1 \leq k \leq N$ we get

$$E_i(u_{ik}) = N \sum_{j=1}^{N} h(u_{ik}u_{ij})u_{ij} = N \sum_{j=1}^{N} \delta_{kj} h(u_{ij})u_{ij} = h(u_{ik}).$$

Thus E_i is surjective and also $E_i \circ E_i = E_i$, since $E_i(b) = b$ for all $b \in E_i(C_\bullet(S_N^+))$, by linearity. □

Proposition 4.3.2: *Let G be a compact quantum group, and H be a dual subgroup of G. Then the unique linear map $E\colon \mathrm{Pol}(G) \to \mathrm{Pol}(H)$ such that*

$$(\mathrm{id} \otimes E)u^x = \begin{cases} u^x & \text{if } x \in \mathrm{Irr}(H), \\ 0 & \text{if } x \in \mathrm{Irr}(G) \setminus \mathrm{Irr}(H) \end{cases}$$

has a unique unital completely positive extension to a map $E_\bullet \colon C_\bullet(G) \to C_\bullet(H)$.

The reduced case follows directly from [Ver04, Proposition 2.2] and the full case follows from [Chi14, Theorem 3.1.]. From now on we denote the map $E_\bullet$ for either the reduced or full case by E_H.

Now we can equip our graph of C^*-algebras with conditional expectations

$$\mathrm{id} \otimes E_k \colon C_\bullet(H) \otimes C_\bullet(S_N^+) \to C_\bullet(H) \otimes L_k$$

and

$$E_H \otimes \mathrm{id} \colon C_\bullet(G) \otimes \mathbb{C}^N \to C_\bullet(H) \otimes \mathbb{C}^N$$

.

Our goal is now to prove that the maximal respectively vertex-reduced fundamental C^*-algebra are isomorphic to the maximal respectively reduced amalgamated free wreath product $C_\bullet(G \wr_{*,H} S_N^+)$.

For the first one, denote by $\mathcal{A} := \pi_1^{\max}(\mathcal{T}_N, (A_q)_{q\in V(\mathcal{T}_N)}, (B_e)_{e\in E(\mathcal{T}_N)}, \mathcal{T}_N)$ its maximal fundamental C^*-algebra relative to the maximal subtree $\mathcal{T}_N$, so that $\mathcal{A}$ is the universal C^*-algebra generated by A_{p_k}, $1 \leq k \leq N$ with the relations $r_{v_k}(a) = s_{v_k}(a)$ for all $a \in C(H) \otimes \mathbb{C}^N$ and all $1 \leq k \leq N$.

Recall $\nu_i \colon C(G) \to C(G)^{*_H N} \subseteq C(G \wr_{*,H} S_N^+)$ denotes the embedding into the i-th copy of $C(G)$ in $C(G)^{*_H N}$, and denote by ν the common restriction of ν_i to $C(H)$. As done in [FT24] we now want to prove $\mathcal{A} \cong C(G \wr_{*,H} S_N^+)$.

Proposition 4.3.3: *Let G be a compact quantum group and $H \subseteq G$ a dual quantum subgroup. There is a unique isomorphism $\pi \colon \mathcal{A} \to C(G \wr_{*,H} S_N^+)$ such that*

$$\begin{cases} \pi(h \otimes u_{ij}) = \nu(h)u_{ij}, & \text{if } h \otimes u_{ij} \in A_{p_0} = C(H) \otimes C(S_N^+) \\ \pi(a \otimes e_j) = \nu_i(a)u_{ij}, & \text{if } a \otimes e_j \in A_{p_i} = C(G) \otimes \mathbb{C}^N \end{cases}$$

for all $1 \leq i, j \leq N$.

Proof. The existence will follow by the universal property of $\mathcal{A}$. Since $\nu_i(a)$ and u_{ij} commute in $C(G \wr_{*,H} S_N^+)$ by definition, we get unique unital *-homomorphisms $\pi_i \colon A_{p_i} \to C(G \wr_{*,H} S_N^+)$ such that $\pi_i(a \otimes e_j) = \nu_i(a)u_{ij}$ for all $a \in C(G)$ and $1 \leq j \leq N$. Moreover define $\pi_0 \colon A_{p_0} \to C(G \wr_{*,H} S_N^+)$ by $\pi_0(h \otimes u_{ij}) = \nu(h)u_{ij}$ for all $h \otimes u_{ij} \in A_{p_0}$. Obviously we have

$$\pi_0(s_{v_i}(h \otimes e_j)) = \pi_0(h \otimes u_{ij}) = \nu(h)u_{ij}$$
$$\pi_i(r_{v_i}(h \otimes e_j)) = \pi_i(h \otimes e_j) = \nu_i(h)u_{ij} = \nu(h)u_{ij},$$

for all $1 \leq i, j \leq N$ and $h \in C(H)$. Note the last step follows since we assume that the restrictions of ν_i for all $1 \leq i \leq N$ on $C(H)$ are the same.

By the universal property of $\mathcal{A}$ we now obtain a unital *-homomorphism fulfilling the conditions we want. The image of π contains all u_{ij} for all $1 \leq i, j \leq N$ and

$$\sum_{j=1}^N \pi_i(a \otimes e_j) = \sum_{j=1}^N \nu_i(a)u_{ij} = \nu_i(a)\sum_{j=1}^N u_{ij} = \nu_i(a)$$

for all $1 \leq i \leq N$, for the last step recall the definition of S_N^+. By this we get surjectivity of π.

Lastly we want to construct an inverse, to show that π is an isomorphism. By the universal property of $C(G \wr_{*,H} S_N^+)$ we obtain a unique unital surjective *-homomorphism $\mu \colon C(G)^{*N} * C(S_N^+) \to C(G \wr_{*,H} S_N^+)$ such that $\mu(\nu_i(a)) = a \otimes 1 \in A_{p_i}$ and $\mu(b) = 1 \otimes b \in A_{p_0}$ for all $1 \leq i \leq N$, $a \in C(G)$ and $b \in C(S_N^+)$. We now want to prove that I from the definition of $C(G \wr_{*,H} S_N^+)$ is contained in $\ker(\mu)$, by this we obtain a map $\rho\colon C(G \wr_{*,H} S_N^+) \to \mathcal{A}$. Indeed, note that

$$\begin{aligned}\mu(\nu_i(a)u_{ij}) = a \otimes u_{ij} &= (a \otimes 1)(1 \otimes u_{ij}) = \mu(\nu_i(a))s_{v_j}(1 \otimes e_j) \\ &= \mu(\nu_i(a))r_{v_j}(1 \otimes e_j) = (a \otimes 1)(1 \otimes e_j) \\ &= (1 \otimes e_j)(a \otimes 1) = r_{v_i}(1 \otimes e_j)\mu(\nu_i(a)) \\ &= s_{v_i}(1 \otimes e_j)\mu(\nu_i(a)) = (1 \otimes u_{ij})\mu(\nu_i(a)) = \mu(u_{ij}\nu_i(a))\end{aligned}$$

for all $1 \leq i, j \leq N$ and $a \in C(G)$. Thus $I \subseteq \ker(\mu)$.

Recall now that $\iota\colon C(H) \to C(G)$ is the faithful *-homomorphism from Definition 3.3.18. Then the following holds

$$\begin{aligned}\mu(\nu_i(\iota(h))) = \iota(h) \otimes 1 = s_{v_i}(h \otimes 1) &= r_{v_i}(h \otimes 1) = h \otimes 1 \\ &= r_{v_j}(h \otimes 1) = s_{v_j}(h \otimes 1) = \iota(h) \otimes 1 = \mu(\nu_j(\iota(h))).\end{aligned}$$

Thus the images of $C(H)$ on μ coincide, and from $I \subseteq \ker(\mu)$ we obtain a unique unital *-homomorphism $\rho\colon C(G \wr_{*,H} S_N^+) \to \mathcal{A}$ factorising μ. Since μ is surjective, also ρ is surjective and we can easily prove that ρ is the inverse of π. Indeed, since the images of $C(H)$ on μ coincide, they also coincide on ρ and we get

$$\begin{aligned}(\rho \circ \pi)(h \otimes u_{ij}) = \rho(\nu(h)u_{ij}) = h \otimes u_{ij}, \\ (\rho \circ \pi)(a \otimes u_{ij}) = \rho(\nu_i(a)u_{ij}) = a \otimes u_{ij},\end{aligned}$$

for all $1 \leq i, j \leq N$, $h \in C(H)$ and $a \in C(G)$. Hence ρ is the inverse of π and thus π is an isomorphism. □

We now proved that $\mathcal{A} \cong C(G \wr_{*,H} S_N^+)$, this will be used later to calculate the K-theory of $C(G \wr_{*,H} S_N^+)$. We now also want to do the same for the reduced case. For this we will need to look at the Haar state.

But let us fix some notation before doing so. Denote by $\mathcal{A}_r$ the vertex-reduced fundamental C^*-algebra of the constructed graph of C^*-algebras with faithful conditional expectations as constructed and view $A_{p_k} \subseteq \mathcal{A}$ for all $0 \leq k \leq N$. We write λ for the canonical surjection of the full to the reduced fundamental C^*-algebra.

By the universal property of $\mathcal{A}$, there exists a unique surjective unital $*$-homomorphism $\lambda'\colon \mathcal{A} \to \mathcal{A}_r$ such that the restrictions fulfil $\lambda'|_{\mathcal{A}_{p_0}} = \lambda_H \otimes \lambda_{S_N^+}$ and $\lambda'|_{\mathcal{A}_{p_k}} = \lambda_G \otimes \mathrm{id}_{\mathbb{C}^N}$ for all $1 \leq k \leq N$. Let $E\colon \mathcal{A} \to C_r(H) \otimes C_r(S_N^+)$ be a GNS-faithful conditional expectation and define $\omega := h_H \otimes h_{S_N^+} \circ E$ such that the restrictions fulfil $\omega|_{A_{p_0}} = h_H \otimes h_{S_N^+}$ and $\omega(c) = 0$ for all reduced operators in A. The state $\omega \in A^*$ is called the *fundamental state.* Let $\lambda\colon C(G \wr_{*,H} S_N^+) \to C_r(G \wr_{*,H} S_N^+)$ be the canonical surjection.

Proposition 4.3.4: *Let G be a compact quantum group and $H \subseteq G$ a dual quantum subgroup. The unique Haar state $h \in C(G \wr_{*,H} S_N^+)^*$ vanishes on reduced operators of the form*

$$a_0\nu_{i_1}(b_1)a_1\nu_{i_2}(b_2)\ldots\nu_{i_n}(b_n)a_n,$$

where $a_k \in C(S_N^+) \subset C(G \wr_{*,H} S_N^+)$*, and* $b_k \in C(G)$ *are such that* $E_H(b_k) = 0$ *for all* k *and if* $i_k = i_{k+1}$*, then* $E_{i_k}(a_k) = 0$*.*

There exists a unique unital **-isomorphism* $\pi_r\colon \mathcal{A}_r \to C_r(G \wr_{*,H} S_N^+)$ *such that* $\lambda \circ \pi = \pi_r \circ \lambda'$*, where* $\pi : \mathcal{A} \to C(G \wr_{*,H} S_N^+)$ *is the isomorphism of Proposition Proposition 4.3.3.*

Proof. Define the state $\widetilde{\omega} := \omega \circ \lambda' \circ \mu \in C(G \wr_{*,H} S_N^+)^*$, where the *-homomorphism $\mu := \pi^{-1} : C(G \wr_{*,H} S_N^+) \to \mathcal{A}$ has been constructed in the proof of the full case in Proposition 4.3.3. Let $\mathcal{C} \subset \mathcal{A}$ be the linear span of $C(S_N^+)$, $\nu(C(H))$, and all reduced operators in $C(G \wr_{*,H} S_N^+)$, note that $\mathcal{C}$ is dense in $\mathcal{A}$ by definition. By construction, the state $\widetilde{\omega}$ satisfies $\widetilde{\omega}|_{C(S_N^+)} = h_{S_N^+}$, $\widetilde{\omega} \circ \nu = h_H$, and $\widetilde{\omega}(c) = 0$ for $c \in C(G \wr_{*,H} S_N^+)$ being a reduced operator. Thus, $\widetilde{\omega}$ satisfies the properties of the state h as stated in the theorem. Hence, $h = \widetilde{\omega}$ by the density of $\mathcal{C}$. We then will show that $\widetilde{\omega}$ is invariant with respect to Δ, hence $\widetilde{\omega}$ is the Haar state. To do this we it suffices to prove that $\widetilde{\omega}$ is invariant on $\mathcal{C}$. For an element $x \in \mathcal{C}$, which is the sum of an element $x_0 \in C(S_N^+) \otimes C(H)$ and reduced operators. By construction of $\widetilde{\omega}$ it suffices to show that $\Delta(x) \in \mathcal{C} \odot C(G \wr_{*,H} S_N^+)$, as it is done in [FT24, Theorem 3.2].

Since then $\widetilde{\omega} = h$ is the Haar state, and since E is GNS faithful and $h_H \otimes h_{S_N^+}$ is faithful on $C_r(H) \otimes C_r(S_N^+)$, we obtain that $\mathcal{A}_r$ is isomorphic to the reduced C^*-algebra $C_r(G \wr_{*,H} S_N^+)$, since it is constructed by the GNS construction with respect to the Haar state. ☐

Overall, we can therefore summarise, where $\mathcal{A}_\bullet$ stands either for the full or the vertex-reduced fundamental C^*-algebra.

Corollary 4.3.5: *Let* G *be a compact quantum group and* $H \subseteq G$ *a dual quantum subgroup. Then* $C_\bullet(G \wr_{*,H} S_N^+)$ *is isomorphic to the vertex-reduced/full fundamental* C^**-algebra* $\mathcal{A}_\bullet$ *of the constructed graph of* C^**-algebras equipped with conditional expectations.*

We will now use this in the next chapter to compute the K-theory of free wreath products with trivial amalgamation.

Chapter V.

K-theory of free wreath products

In this chapter we want to compute the K-theory of free wreath products of compact quantum groups with the quantum symmetric group S_N^+. To do so we need to prove the 6-term exact sequences for the KK-theory of a fundamental C^*-algebra and after this we can directly compute the K-theory of free wreath products. Before we want to have a look at the vertex-reduced free product and the vertex-reduced HNN-extension. This chapter is based primarily on the work of Fima on graph of C^*-algebras [Fim13; FF14; FP16; FG18; FG20; FT24].

1. Vertex-reduced free product and HNN-extension

In [FG20; Fim13], Fima and Germain proved the KK-equivalence of the full free amalgamated products with the *vertex-reduced* free amalgamated product, respectively the KK-equivalence of the full HNN-extension with the vertex-reduced one. To prove this they constructed six-term exact sequences. In [FG18] Fima and Germain generalised it in the sense of graph of C^*-algebras and fundamental C^*-algebras.

Indeed as we saw in Example 4.1.7, for the graph consisting of two vertices and one geometric edge, we obtain as full fundamental C^*-algebra the full amalgamated free product, analogously if one equip this setting with conditional expectations, which are not GNS-faithfully, then one obtain as vertex-reduced fundamental C^*-algebra the *vertex-reduced amalgamated free product.* Note that there is also another possibility if the conditional expectations are GNS-faithful, the *edge-reduced amalgamated free product*, introduced by Voiculescu. However it turns out, that this construction is too “small” in some sense. For the HNN-extension the same can be done.

For this let us introduce their vertex-reduced counterpart, before introducing *Serre's dévissage process.*

We will begin with the vertex-reduced HNN-extension. To do so we will modify the definition, such that we can “more naturally” define the vertex-reduced HNN-extension. Recall Example 4.1.7.

Remark 5.1.1 (HNN-extension): Let A, B be unital C^*-algebras, let $\pi_k\colon B \to A$ be unital faithful *-homomorphism and let $E_k\colon A \to B$ be a unital completely positive map such that $E_k \circ \pi_k = \mathrm{id}_B$ for all $k = -1, 1$. The full HNN-extension is the universal C^*-algebra generated by A and a unitary u such that $u\pi_{-1}(b)u^* = \pi_1(b)$ for all $b \in B$. We denote it by $\mathrm{HNN}(A, B, \pi_1, \pi_{-1})$.

Definition 5.1.2 (Vertex-reduced HNN-extension): The vertex-reduced HNN-extension C is the unique, up to isomorphism, unital C^*-algebra satisfying the following properties:

(i) There exists a unital *-homomorphism $\rho\colon A \to C$ and a unitary $u \in C$ such that $u\rho(\pi_{-1}(b))u^* = \rho(\pi_1(b))$ for all $b \in B$ and C is generated by $\rho(A)$ and u.

(ii) There exists a GNS-faithful unital completely positive map $E\colon C \to A$ such that $E \circ \rho = \mathrm{id}_A$ and $E(x) = 0$ for all $x \in C$ of the form $x = \rho(a_0)u^{\epsilon_1} \cdots u^{\epsilon_n}\rho(a_n)$ where $n \geq 1$, $a_k \in A$ and $\epsilon_k \in \{-1, 1\}$ are such that, for all $1 \leq k \leq n-1$, $\epsilon_{k+1} = -\epsilon_k$ then $E_{-\epsilon_k}(a_k) = 0$.

(iii) If D is a unital C^*-algebra with a unital *-homomorphism $\nu\colon A \to D$, a unitary $v \in D$ and a GNS-faithful unital completely positive map $E'\colon D \to A$ such that

- $v\nu(\pi_{-1}(b))v^* = \nu(\pi_1(b))$ for all $b \in B$ and D is generated by $\nu(A)$ and v.
- $E' \circ \nu = \mathrm{id}_A$ and $E'(x) = 0$ for all $x \in D$ of the form $x = \nu(a_0)v^{\epsilon_1} \cdots v^{\epsilon_n}\nu(a_n)$ with $n \geq 1$, $\epsilon_k \in \{-1, 1\}$, $a_k \in A$ such that, for all $1 \leq k \leq n-1$ one has $\epsilon_{k+1} = -\epsilon_k$ then $E_{-\epsilon_k}(a_k) = 0$.

Then there exists a unique unital $*$-homomorphism $\tilde{\nu} : C \to D$ such that $\tilde{\nu} \circ \rho = \nu$ and $\tilde{\nu}(u) = v$. Moreover, $E' \circ \tilde{\nu} = E$. We denote this C^*-algebra by $\mathrm{HNN}_{\mathrm{vert}}(A, B, \pi_1, \pi_{-1})$.

The construction of the vertex-reduced amalgamated free product is a little bit more technical. For this recall notations from Section 2 of Chapter 4.

For two C^*-algebras A_1, A_2 and a common C^*-subalgebra B, we will denote $A_f := A_1 *_B A_2$ for the full free amalgamated product. Moreover we assume that we have conditional expectations $E_k\colon A_k \to B$ for $k = 1, 2$. We write

$$A_k^\circ = \{a \in A_k \mid E_k(a) = 0\},$$

and we denote by (K_k, ρ_k, ν_k) the GNS-construction as in Construction 4.2.1, and recall that K_k° is the orthogonal complement of $\eta_k B$ in K_k. Now Let

$$I := \{(i_1, \ldots, i_n) \in \{1, 2\}^n \mid n \geq 1 \text{ and } i_{k-1} \neq i_k \text{ for all } 2 \leq k \leq n\}$$

and define for $\underline{i} = (i_1, \ldots, i_n) \in I$ the Hilbert A_{i_n}-module

$$H_{\underline{i}} := \begin{cases} K_{i_1} \otimes K_{i_2}^\circ \otimes \ldots \otimes K_{i_{n-1}}^\circ \otimes A_{i_n} & , \text{ for } n \geq 3, \\ K_{i_1} \otimes A_{i_2} & , \text{ for } n = 2, \\ A_{i_1} & , \text{ for } n = 1. \end{cases}$$

The left action of A_{i_1} on $H_{\underline{i}}$ is given by

$$\lambda_{\underline{i}}\colon A_{i_1} \to \mathcal{B}_{A_{i_n}}(H_{\underline{i}}), \quad \lambda_{\underline{i}} = \begin{cases} \rho_{i_1} \otimes \mathrm{id} & , \text{ for } n \geq 2, \\ L_{A_{i_1}} & , \text{ for } n = 1, \end{cases}$$

where $L_{A_{i_1}}$ is the left multiplication operator of A_{i_1}.

Moreover we set for $k, l \in \{1, 2\}$

$$I_{k,l} := \{(i_1, \dots, i_n) \in I \mid i_1 = k,\ i_n = l\}$$

and

$$H_{k,l} = \bigoplus_{\underline{i} \in I_{k,l}} H_{\underline{i}}, \quad \lambda_{k,l} = \bigoplus_{\underline{i} \in I_{k,l}} \lambda_{\underline{i}}\colon A_k \to \mathcal{B}_{A_l}(H_{k,l}).$$

We now define unitaries $u_{k,l}$ on $\mathcal{B}_{A_l}(H_{k,l}, H_{k,l})$.

Let $\underline{i} = (i_1, \dots, i_n) \in I$, with $i_1 = k$ and $i_l = l$. For $\xi \in H_{\underline{i}}$ we define $u_{k,l}\xi \in H_{\underline{i}}$ in the following way

- If $n \geq 2$, write $\underline{i} = (k, \underline{i}')$, where $\underline{i}' = (i_2, \dots, i_n) \in I_{k,l}$. For $\xi = \rho_k(a)\eta_k \otimes \xi'$, with $a \in A_k$ and $\xi' \in H_{\underline{i}'}$, we define

$$u_{k,l}\xi = \begin{cases} \eta_k \otimes \xi', & \text{if } E_k(a) = 0, \\ \lambda'_k(a)\xi', & \text{if } a \in B. \end{cases}$$

- If $n = 1$ then $k = l$ and $\underline{i}'$ is empty and $\xi \in A_l = H_{k,l}$. We define $u_{k,l}\xi = \lambda'_k(\xi) = \eta_k \otimes \xi$.

For all $k, l \in \{1, 2\}$, the operator $u_{k,l}$ commutes with the right actions of A_l on $H_{k,l}$ and $H_{\overline{k},l}$ and extends to a unitary operator, which we will still denote by $u_{k,l} \in \mathcal{B}_{A_l}(H_{k,l}, H_{T,k,l})$ such that $u^*_{k,l} = u_{\overline{k},l}$.

We now can define the *k-vertex-reduced amalgamated free product.*

Definition 5.1.3 (*k*-vertex-reduced): Let $k \in \{1, 2\}$. The *k-vertex-reduced amalgamated free product* is the C^*-subalgebra $A_{v,k} \subset \mathcal{B}_{A_k}(H_{k,k})$ generated by $\lambda_{k,k}(A_k) \cup u_{\overline{k},k}\lambda_{\overline{k},k}(A_{\overline{k}})u_{k,k} \subset \mathcal{B}_{A_k}(H_{k,k})$.

By definition of $u_{k,l}$ we also have

$$u^*_{k,l}\lambda_{\overline{k},l}(b)u_{k,l} = \lambda_{k,l}(b)$$

for all $b \in B$, which imply the existence of a unique unital $*$-homomorphism

$$\pi_k : A_f \to A_{v,k}, \quad \pi_k(a) = \begin{cases} \lambda_{k,k}(a), & \text{if } a \in A_k, \\ u^*_{k,k}\lambda_{\overline{k},k}(a)u_{k,k}, & \text{if } a \in A_{\overline{k}}. \end{cases}$$

Definition 5.1.4 (Vertex-reduced free product): The *vertex-reduced amalgamated free product* is the C^*-algebra obtained by separation and completion of the full free amalgamated free product $A_f = A_1 *_B A_2$ with respect to the C^*-seminorm $\|\cdot\|$ on A_f defined by

$$\|x\| := \max\{\|\pi_1(x)\|, \|\pi_2(x)\|\},$$

for $x \in A_f$. We denote $A_1 \overset{v}{*}_B A_2$ for the vertex-reduced free product.

Note that the details of the construction are not that much relevant for this thesis, however it is necessary to give a definition. For more details for the vertex-reduced free product and its properties, see for instance [FG20].

Now we want to quickly describe Serre's dévissage process for graphs of groups, which can be done analogue for graph of C^*-algebras. This technique will be an important idea to prove the six-term exact sequence in Theorem 5.2.1.

Remark 5.1.5 (Serre's dévissage technique): To obtain the fundamental group of a graph of groups in Bass-Serre theory, Serre described in [Ser77] the so called *dévissage technique*, to easily compute the fundamental group. Doing this he proved that fundamental groups of graph of groups are inductive limit of iterations of amalgamated free products and HNN extensions.

Start with a connected (finite graph) $\mathcal{G}$ and remove an edge e and $\overline{e}$, and obtain a new graph $\mathcal{G}'$. If $\mathcal{G}'$ is still connected, then the fundamental group is a HNN extension of the fundamental group of the remaining graph. Otherwise if $\mathcal{G}'$ is disconnected, then the fundamental group is the free amalgamated product of the fundamental groups of the two connected components.

The same can be proven for graph of C^*-algebras, see [FF14; FG18].

2. 6-term exact sequences for KK-theory of a fundamental C^*-algebra

In the following setting we assume that we have conditional expectations, which do not need to be GNS-faithful.

Recall that a sequence

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} G_4 \xrightarrow{f_4} \dots$$

of groups $G_1, G_2, \dots$ and group homomorphisms $f_1, f_2, \dots$ is called to be *exact at G_i* if $\ker(f_{i+1}) = \operatorname{im} f_i$. The sequence is called *exact* if it is exact at every G_i. Note that every exact sequence is a *chain complex*, since $f_{i+1} \circ f_i = 0$.

Our main idea to prove the following theorem will be mainly to use Serre's dévissage process and induction on the number of edges. Also we need to use the six-term exact sequences for the free product and the HNN-extension.

Theorem 5.2.1: *For a graph of separable C^*-algebras $(\mathcal{G}, (A_q)_{q\in V(\mathcal{G})}, (B_e)_{e\in E(\mathcal{G})})$, where we denote by $P_\bullet$ either the full or vertex-reduced fundamental C^*-algebra, we have the following two 6-term exact sequences.*

$$\begin{array}{ccccc}
\bigoplus_{e\in E^+(\mathcal{G})} KK^0(C,B_e) & \xrightarrow{\sum s_e^*-r_e^*} & \bigoplus_{p\in V(\mathcal{G})} KK^0(C,A_p) & \longrightarrow & KK^0(C,P_\bullet) \\
\uparrow & & & & \downarrow \\
KK^1(C,P_\bullet) & \longleftarrow & \bigoplus_{p\in V(\mathcal{G})} KK^1(C,A_p) & \xleftarrow[\sum s_e^*-r_e^*]{} & \bigoplus_{e\in E^+(\mathcal{G})} KK^1(C,B_e),
\end{array}$$

and

$$\begin{array}{ccccc}
\bigoplus_{e\in E^+(\mathcal{G})} KK^0(B_e,C) & \xleftarrow[\sum s_{e*}-r_{e*}]{} & \bigoplus_{p\in V(\mathcal{G})} KK^0(A_p,C) & \longleftarrow & KK^0(P_\bullet,C) \\
\downarrow & & & & \uparrow \\
KK^1(P_\bullet,C) & \longrightarrow & \bigoplus_{p\in V(\mathcal{G})} KK^1(A_p,C) & \xrightarrow{\sum s_{e*}-r_{e*}} & \bigoplus_{e\in E^+(\mathcal{G})} KK^1(B_e,C).
\end{array}$$

We will only prove the exactness of the first diagram, since for the second case it is more or less the same, and does not bring any new aspects.

Before we can prove the theorem, we need to construct the boundary maps, i.e. the vertical maps. Since this is very technical we will skip the details and try to give a quick idea how to construct these maps. Details can be found in [FG18].

As done in [FG18] one can construct a unique unital completely positive map $\mathbb{E}_{A_p}\colon P_{\text{vert}} \to A_p$ for all $p \in V(\mathcal{G})$ such that $\mathbb{E}_{A_p} \circ \lambda(a) = a$ for all $a \in A_p$ and $p \in V(\mathcal{G})$, where $\lambda\colon P \to P_{\text{vert}}$ is the canonical surjection.

Then we can define a unital completely positive map $\mathbb{E}_e = E_e^r \circ \mathbb{E}_{A_{r(e)}}\colon P_{\text{vert}} \to B_e^r$, and we will denote the GNS-construction by (K_e, ρ_e, η_e) for all $e \in E(\mathcal{G})$. Setting $\mathcal{R}_e \subseteq K_e$ as the Hilbert B_e^r-submodule of K_e of words ending with e, we take the projection Q_e from K_e into $\mathcal{R}_e$. One then can prove that

$$[Q_e \rho_e(\lambda(a))] \in \mathcal{K}_{B_e^r}(K_e)$$

for all $a \in P$, where $[\cdot,\cdot]$ denotes the commutator. We then define $V_e = 2Q_e - 1 \in \mathcal{B}_{B_e^r}(K_e)$, which fulfils $V_e^2 = 1$, $V_e^* = V_e$ and for all $x \in P_{\text{vert}}$ we have $[V_e \rho_e(x)] \in \mathcal{K}_{B_e^r}(K_e)$. Recalling the definition of Kasparov modules, and KK-groups, we therefore have an KK-element $y_e^{\mathcal{G}} \in KK^1(P_{\text{vert}}, B_e^r)$. Then define

$$x_e^{\mathcal{G}} = y_e^{\mathcal{G}} \otimes_{B_e^r} [r_e^{-1}] \in KK^1(P_{\text{vert}}, B_e)$$

and

$$z_e^{\mathcal{G}} = [\lambda] \otimes_{P_{\text{vert}}} x_e^{\mathcal{G}} \in KK^1(P, B_e).$$

One then can define the boundary maps $\gamma_e^{\mathcal{G}}$ by

$$\gamma_e^{\mathcal{G}}\colon KK^*(C,P_\bullet)\to KK^{*+1}(C,B_e),\quad y\mapsto\begin{cases} y\otimes_P z_e^{\mathcal{G}} & \text{, in the full case,}\\ y\otimes_{P_{\mathrm{vert}}} x_e^{\mathcal{G}} & \text{, in the vertex-red. case.}\end{cases}$$

Later we will write x_e for $x_e^{\mathcal{G}}$ and z_e for $z_e^{\mathcal{G}}$.

Let us fix some notation. We will denote P for either the full or the vertex-reduced fundamental C^*-algebra of $\mathcal{G}$. Taking subgraphs $\mathcal{H}\subseteq\mathcal{G}$ we will denote $P^{\mathcal{H}}$ for the full or vertex-reduced fundamental C^*-algebra of $\mathcal{H}$. If $\mathcal{G}$ has after removing the geometric edge e_0 two connected components, we will write V_1 and V_2 for the two sets of vertices and E_1 and E_2 for the sets of edges. Obviously they are both disjoint. We write $\pi_v^{\mathcal{G}}$ for the canonical *-homomorphism $A_v\to P^{\mathcal{H}}$ for any subgraph $\mathcal{H}\subseteq\mathcal{G}$.

For all the following propositions, where we want to show that step by step, that we have indeed exactness, we will use Serre's dévissage technique, see Remark 5.1.5. We choose a positive edge e_0 of the graph $\mathcal{G}$ and remove e_0 and $\overline{e_0}$. Either it is still a connected graph $\mathcal{G}_0$ and the fundamental C^*-algebra is a HNN-extension (Case 1) or it is disconnected in two graphs $\mathcal{G}_1,\mathcal{G}_2$ and thus we have a amalgamated free product (Case 2).

Proposition 5.2.2: *We have the exactness of*

$$\bigoplus_{e\in E^+(\mathcal{G})}KK^0(C,B_e)\xrightarrow{\sum s_e^*-r_e^*}\bigoplus_{p\in V(\mathcal{G})}KK^0(C,A_p)\xrightarrow{\sum\pi_p^*}KK^0(C,P_\bullet)\ .$$

Proof. We will look at the two cases mentioned above.

Case 1: P is a HNN-extension of $P^{\mathcal{G}_0}$ and B_{e_0}. The set of vertices of $\mathcal{G}_0$ and $\mathcal{G}$ are by assumption the same, and we may assume that $p_0=s(e_0)=r(e_0)$, by identifying $s(p_0)=r(p_0)$ in spirit of Example 4.1.7. Let $x\in\bigoplus_p KK^0(C,A_p)$ with $x=\oplus_p x_p$ such that $x\in\ker(\sum_p\pi_p^*)$, i.e. $\sum_p\pi_p^*(x_p)=0$ and if $y=\sum_v\pi_v^{0*}(x_v)$, then we have $\pi_{\mathcal{G}_0}(y)=0$. Then, the long exact sequence for P seen as an HNN extension, proven in [Fim13], implies that there exists $y_0\in KK^0(C,B_{e_0})$ such that $(\pi_{v_0}\circ s_{e_0})^*(y_0)-(\pi_{v_0}\circ r_{e_0})^*(y_0)=y=\sum_v\pi_v^{0*}(x_v)$. Hence,

$$\sum_v\pi_v^{0*}(\oplus_{v\neq v_0}x_v\oplus(x_{v_0}-s_{e_0}^*(y_0)+r_{e_0}^*(y_0))=0.$$

Using the exactness for P_0 as $\mathcal{G}_0$ has one edge less, we get that there exists for any $e\neq e_0$ a y_e such that

$$\sum_{e\neq e_0}s_e^*(y_e)-r_e^*(y_e)=\oplus_{v\neq v_0}x_v\oplus(x_{v_0}-s_{e_0}^*(y_0)+r_{e_0}^*(y_0)).$$

Thus,

$$\sum_{e\neq e_0}s_e^*(y_e)-r_e^*(y_e)+s_{e_0}^*(y_0)-r_{e_0}^*(y_0)=x.$$

This proves the exactness in this case.

Case 2: Now P is the amalgamated free product of $P_1 = P_{\mathcal{G}_1}$ and $P_2 = P_{\mathcal{G}_2}$ over B_{e_0}. Denote by the map $\pi_v^i\colon A_v \to P_i$ the canonical embedding and write $v_1 = s(e_0)$ and $v_2 = r(e_0)$.

Now let $x = \oplus x_v$ be in $\bigoplus_p KK^0(C, A_p)$ such that $\sum_v \pi_v^*(x_v) = 0$ and moreover let $x_i = \oplus_{v\in V_i} {\pi_v^i}^*(x_v)$. Obviously we have $\pi_{\mathcal{G}_1}^*(x_1) + \pi_{\mathcal{G}_2}^*(x_2) = 0$. Then, the long exact sequence for P seen as an amalgamated free product, as done in [FP16], gives us an element $y_0 \in KK^0(C, B_{e_0})$ such that

$$(\pi_{v_1}^1 \circ s_{e_0})^*(y_0) - (\pi_{v_2}^2 \circ r_{e_0})^*(y_0) = x_1 \oplus x_2.$$

We then define $\bar{x}_1 = \oplus_{v\in V_1} x_v - s_{e_0}^*(y_0)$ and $\bar{x}_2 = \oplus_{v\in V_2} x_v + r_{e_0}^*(y_0)$. We have $\sum_{v\in V_i} {\pi_v^i}^*(\bar{x}_i) = 0$ for $i = 1, 2$. As the graphs $\mathcal{G}_1, \mathcal{G}_2$ have strictly less edges than $\mathcal{G}$, by inductive arguments on the number of edges there exists for any $e \neq e_0$ a $y_e \in KK^0(C, B_e)$ such that

$$\bar{x}_1 \oplus \bar{x}_2 = \sum_{e\neq e_0} s_e^*(y_e) - r_e^*(y_e).$$

Hence we obtain

$$x = \sum_{e\neq e_0} s_e^*(y_e) - r_e^*(y_e) + s_{v_0}^*(y_0) - r_{v_0}^*(y_0).$$

Hence also in this case we obtain exactness. □

Proposition 5.2.3: *We have the exactness of*

$$\bigoplus_{p\in V(\mathcal{G})} KK^0(C, A_p) \xrightarrow{\sum \pi_p^*} KK^0(C, P_\bullet) \xrightarrow{\oplus\gamma_e} \bigoplus_e KK^1(C, B_e).$$

Proof. **Case 1:** Let $x \in KK^0(C, P)$ such that $\gamma_e^{\mathcal{G}}(x) = 0$ for any edge e, in particular for e_0. Using the long exact sequence for P seen as an HNN-extension as in [Fim13], and since $\gamma_{e_0}^{\mathcal{G}}(x) = 0$ we get that there exists $x_0 \in KK^0(C, P_0)$ such that $\pi_{\mathcal{G}_0}^*(x_0) = x$. For any edges $e \neq e_0$, one then has $\gamma_e^{\mathcal{G}_0}(x_0) = \gamma_e^{\mathcal{G}}(\pi_{\mathcal{G}_0}^*(x_0)) = 0$. Hence by inductive arguments on the number of edges there exists for any $v \in V(\mathcal{G}_0) = V(\mathcal{G})$ an element $y_v \in KK^0(C, A_v)$ such that $\sum_v {\pi_v^0}^*(y_v) = x_0$. Hence

$$x = \sum_v (\pi_{\mathcal{G}_0} \circ \pi_v^0)^*(y_v) = \sum_v \pi_v^*(y_v).$$

Case 2: Using that P is the free amalgamated product of P_1 and P_2 over B_{e_0}, we get an element $x_i \in KK^0(C, P_i)$ for $i = 1, 2$ such that $x = \pi_{\mathcal{G}_1}^*(x_1) + \pi_{\mathcal{G}_2}^*(x_2)$. For any edge e of $\mathcal{G}_i$, $i = 1, 2$, we have

$$\gamma_e^{\mathcal{G}_i}(x_i) = \gamma_e^{\mathcal{G}}(\pi_{\mathcal{G}_i}^*(x_i)) = \gamma_e^{\mathcal{G}}(x) - \gamma_e^{\mathcal{G}}(\pi_{\mathcal{G}_j}^*(x_j)) \text{ for } j \neq i.$$

But since e is not an edge of $\mathcal{G}_j$, we have $\gamma_e^{\mathcal{G}} \circ \pi_{\mathcal{G}_j}^* = 0$. Hence $\gamma_e^{\mathcal{G}_i}(x_i) = 0$. Again by induction we get for any vertex of $V_1 \cup V_2 = V(\mathcal{G})$ an element $y_v \in KK^0(C, A_p)$ such that $x_i = \sum_{p \in V_i} \pi_v^{i\,*}(y_v)$ for $i = 1, 2$. Thus we obtain $x = \sum_v \pi_v^*(y_v)$. □

Proposition 5.2.4: *We have the exactness of*

$$KK^0(C, P_\bullet) \xrightarrow{\oplus \gamma_e} \bigoplus_e KK^1(C, B_e) \xrightarrow{\sum s_e^* - r_e *} \bigoplus_p KK^1(C, A_p)$$

Proof. **Case 1:** Let $x = \oplus_e x_e$ such that $\sum_e s_e^*(x_e) - r_e^*(x_e) = 0$. Then for the distinguished vertex v_0, one has

$$(\pi_{v_0}^0)^*(s_{e_0}^*(x_{e_0})) - (\pi_{v_0}^0)^*(r_{e_0}^*(x_{e_0})) = -\sum_{e \neq e_0} (\pi_{v_0}^0)^*(s_e^*(x_{e_0})) - (\pi_{v_0}^0)^*(r_e^*(x_{e_0})).$$

Since e is an edge of $\mathcal{G}_0$, and s_e and r_e are conjugated by a unitary of P_0, we have that their difference is 0 in any KK-group. Thus

$$(\pi_{v_0}^0)^*(s_{e_0}^*(x_{e_0})) - (\pi_{v_0}^0)^*(r_{e_0}^*(x_{e_0})) = 0.$$

Using the long exact sequence for P as an HNN-extension, see [Fim13], we get an element $y_0 \in KK^0(C, P)$ such that $\gamma_{e_0}^{\mathcal{G}}(y_0) = x_{e_0}$. Now set $\bar{x}_e = x_e - \gamma_e^{\mathcal{G}}(y_0)$ for any $e \neq e_0$. Then

$$\begin{aligned}
\sum_{e \neq e_0} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) &= \sum_{e \neq e_0} s_e^*(x_e) - r_e^*(x_e) - \sum_e s_e^*(\gamma_e^{\mathcal{G}}(y_0)) - r_e^*(\gamma_e^{\mathcal{G}}(y_0)) \\
&\quad + s_{e_0}^*(\gamma_{e_0}^{\mathcal{G}}(y_0)) - r_{e_0}^*(\gamma_{e_0}^{\mathcal{G}}(y_0)) \\
&= \sum_e s_e^*(x_e) - r_e^*(x_e),
\end{aligned}$$

by the properties of the boundary maps. Hence we get $\sum_{e \neq e_0} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) = 0$. Again by induction there exists an element $\bar{y}_1 \in KK^0(C, P_0)$ such that for all $e \neq e_0$, we have $\gamma_e^{\mathcal{G}_0}(y_1) = \bar{x}_e$. Set $y_1 = \pi_{\mathcal{G}_0}^*(\bar{y}_1) \in KK^0(C, P)$. Then it holds

$$\gamma_{e_0}^{\mathcal{G}}(y_0 + y_1) = x_0 + \gamma_{e_0}^{\mathcal{G}} \circ \pi_{\mathcal{G}_0}^*(\bar{y}_1).$$

But since e_0 is not an edge of $\mathcal{G}_0$ we have $\gamma_{e_0}^{\mathcal{G}} \circ \pi_{\mathcal{G}_0}^* = 0$. Thus $\gamma_{e_0}^{\mathcal{G}}(y_0 + y_1) = x_0$.

For $e \neq e_0$ we have $\gamma_e^{\mathcal{G}}(y_0 + y_1) = \gamma_e^{\mathcal{G}}(y_0) + \bar{x}_e$ as $\gamma_e^{\mathcal{G}_0} = \gamma_e^{\mathcal{G}} \circ \pi_{\mathcal{G}_0}^*$. It follows that $\gamma_e^{\mathcal{G}}(y_0 + y_1) = x_e$ and by this the exactness in this case.

Case 2: Note that for any positive edge e, if $s(e) \in V_1$ then either $e \in E_1$ or $e = e_0$ and if $r(e) \in V_2$ then $e \in E_2$.

Let $x = \oplus_e x_e$ such that $\sum_e s_e^*(x_e) - r_e^*(x_e) = 0$. We can rewrite it as

$$\sum_{e \in E_1^+} s_e^*(x_e) - r_e^*(x_e) + s_{e_0}^*(x_{e_0}) = 0 \in \bigoplus_{p \in V_1} KK^1(C, A_p)$$

and

$$\sum_{e\in E_2^+} s_e^*(x_e) - r_e^*(x_e) - r_{e_0}^*(x_{e_0}) = 0 \in \bigoplus_{p\in V_2} KK^1(C, A_p).$$

We have $\pi_{v_1}^1(x_{e_0}) = -\sum_{e\in E_1^+}(\pi_{s(e)}^1 \circ s_e)^*(x_e) - (\pi_r^1(e)\circ r_e)^*(x_e)$. Since s_e and r_e are conjugated in P_1 because e is an edge of $\mathcal{G}_1$, we obtain that it must be 0. Analogously we have $\pi_{v_2}^2(x_{e_0}) = 0$. Hence by using the long exact sequence for P as a free product of P_1 and P_2, there is an element $y_0 \in KK^0(C, P)$ such that $\gamma_{e_0}^{\mathcal{G}}(y_0) = x_{e_0}$. Now for all edges $e \neq e_0$ we set $\bar{x}_e = x_e - \gamma_e^{\mathcal{G}}(y_0)$. Then

$$\sum_{e\in E_1^+} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) = \sum_{e\in E_1^+} s_e^*(x_e) - r_e^*(x_e) - \left(\sum_{e\in E_1^+} s_e^*\circ\gamma_e^{\mathcal{G}}(y_0) - r_e^*\circ\gamma_e^{\mathcal{G}}(y_0)\right).$$

By the properties of the boundary map $\gamma_e^{\mathcal{G}}$ we have that

$$\sum_{e\in E_1^+} s_e^*\circ\gamma_e^{\mathcal{G}} + s_{e_0}^*\circ\gamma_{e_0}^{\mathcal{G}} - \sum_{e\in E_1^+} r_e^*\circ\gamma_e^{\mathcal{G}} = 0$$

using the fact that the sequence is a chain complex (see also at the begin of the proof of Theorem 5.2.1). Hence we obtain

$$\sum_{e\in E_1^+} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) = \sum_{e\in E_1^+} s_e^*(x_e) - r_e^*(x_e) + s_{e_0}^*(x_{e_0}) = 0.$$

In the same way we get $\sum_{e\in E_2^+} s_e^*(\bar{x}_e) - r_e^*(\bar{x}_e) = 0$. Therefore by induction again, there exists for $i = 1, 2$, an element $y_i \in KK^0(C, P_i)$ such that for all e in E_i^+, $\gamma_e^{\mathcal{G}_i}(y_i) = \bar{x}_e$.

We now set $y = y_0 + \pi_{\mathcal{G}_1}(y_1) + \pi_{\mathcal{G}_2}(y_2)$ in $KK^0(C, P)$. Then by construction we have

$$\gamma_{e_0}^{\mathcal{G}}(y) = x_{e_0} + \gamma_{e_0}^{\mathcal{G}}\circ\pi_{\mathcal{G}_1}^*(y_1) + \gamma_{e_0}^{\mathcal{G}}\circ\pi_{\mathcal{G}_2}^*(y_2) = x_{e_0}$$

as $\gamma_{e_0}^{\mathcal{G}}\circ\pi_{\mathcal{G}_i} = 0$ since e_0 is not an edge of $\mathcal{G}_1$ and $\mathcal{G}_2$.

For $e \in E_1$ we have

$$\gamma_e^{\mathcal{G}}(y) = \gamma_e^{\mathcal{G}}(y_0) + \gamma_e^{\mathcal{G}_1}(y_1) + 0$$

as e is not an edge of $\mathcal{G}_2$. Hence $\gamma_e^{\mathcal{G}}(y) = \gamma_e^{\mathcal{G}}(y_0) + \bar{x}_e = x_e$. The same is true for an edge in E_2.

This completes the proof. □

Proof (of Theorem 5.2.1). By Bott-periodicity for KK-theory, it is equivalent to prove that the following chain complex is exact

$$\longrightarrow \bigoplus_e KK^*(C, B_e) \xrightarrow{\sum_e s_e^*-r_e^*} \bigoplus_v KK^*(C, A_v) \xrightarrow{\sum_v \pi_v^*} KK^*(C, P) \xrightarrow{\oplus_e\gamma_e} \bigoplus_e KK^{*+1}(C, B_e) \longrightarrow$$

Note that at the beginning of the proof of [FG18, Theorem 4.1] it is proven that this is indeed a chain complex. In particular one has to use properties of the boundary maps, we did not mention.

The exactness of the chain complex were done in the last propositions. □

The following corollary out of Theorem 5.2.1 is also proved in [FG18].

Let $(\mathcal{G}, (A_p), (B_e), (s_e))$ and $(\mathcal{G}, (A'_p), (B'_e), (s'_e))$ be graphs of unital C^*-algebras with conditional expectations E_e^s respectively $(E_e^s)'$. We assume that there are unital *-homomorphism $\nu_p\colon A_p \to A'_p$ and $\nu_e\colon B_e \to B'_e$ such that $\nu_e = \nu_{\overline{e}}$ and $\nu_{s(e)} \circ s_e = s'_e \circ \nu_e$. Denoting P and P' for the full fundamental C^*-algebras with unitaries u_e and u'_e, we obtain by using the relations for the maps ν_e and ν_p and the universality of P, respectively P', that there exists a unique unital *-homomorphism $\nu\colon P \to P'$ such that

$$\nu\big|_{A_p} = \nu_p, \quad \nu(u_e) = u'_e \text{ for all } p \in V(\mathcal{G}), e \in E(\mathcal{G}).$$

Corollary 5.2.5: *If the maps ν_p, ν_e are K-equivalences such that*

$$(E_e^s)' \circ \nu_{s(e)} = \nu_{s(e)} \circ E_e^s$$

for all $p \in V(\mathcal{G})$ and $e \in E(\mathcal{G})$, then $\nu\colon P \to P'$ is a K-equivalence.

Recall the *Five Lemma*: Let $A, A', B, B', \dots, E, E'$ be abelian groups and we have the following diagram

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}.$$

If the rows are exact and every square commutes, then $C \to C'$ is an isomorphism.

Note that $\nu\colon P \to P'$ is a K-equivalence if and only if $KK(D, P)$ is isomorphic to $KK(D, P')$ and $KK(P, D)$ is isomorphic to $KK(P', D)$ via the by ν induced map for all unital C^*-algebras D.

Proof (of Corollary 5.2.5). We will use Theorem 5.2.1 and the Five Lemma mentioned above. Recall the established maps in the 6-term exact sequences.

By the first 6-term exact sequence we have the following diagram with exact rows

$$\begin{array}{ccccccccc} \bigoplus_e KK(D, B_e) & \to & \bigoplus_p KK(D, A_p) & \to & KK(D, P) & \to & \bigoplus_e KK^1(D, B_e) & \to & \bigoplus_p KK^1(D, A_p) \\ \downarrow \bigoplus \cdot\otimes[\nu_e] & & \downarrow \bigoplus \cdot\otimes[\nu_p] & & \downarrow \cdot\otimes[\nu] & & \downarrow \bigoplus \cdot\otimes[\nu_e] & & \downarrow \bigoplus \cdot\otimes[\nu_p] \\ \bigoplus_e KK(D, B'_e) & \to & \bigoplus_p KK(D, A'_p) & \to & KK(D, P') & \to & \bigoplus_e KK^1(D, B'_e) & \to & \bigoplus_p KK^1(D, A'_p) \end{array}.$$

We need to prove that for any unital C^*-algebra D the squares are commutative. The first and last square are obviously since we assumed, that $\nu_{s(e)} \circ s_e = s'_e \circ \nu_e$.

The second square from left is also obvious, since by universality of ν we have that $\nu \circ \iota = \iota' \circ \nu_p$ for all $p \in V(\mathcal{G})$, where ι, ι' are the canonical embeddings $A_p \subseteq P$ respectively $A'_p \subseteq P'$.

We are done by proving that the third square is commutative. The commutativity of the square is equivalent to the equality $z_e \otimes [\nu_e] = [\nu] \otimes z'_e \in KK^1(P, B'_e)$ where $z_e \in KK^1(P, B_e)$ and $z'_e \in KK^1(P', B'_e)$ as constructed for the proof of Theorem 5.2.1. By the assumption $(E^s_e)' \circ \nu_{s(e)} = \nu_{s(e)} \circ E^s_e$, which gives us an Hilbert C^*-module isomorphism $K_e \otimes B'_e \cong K'_e$, where K_e, K'_e are the Hilbert C^*-modules we construct for the construction of the boundary maps above. Finally, this implement an isomorphism between the Kasparov modules represented by $z_e \otimes [\nu_e]$ and $[\nu] \otimes z'_e$.

Thus by the Five Lemma, we have $KK(D, P) \cong KK(D, P')$ for all unital C^*-algebras. Analogously one shows the same for $KK(P, D) \cong KK(P', D)$ while using the second six-term exact sequence. Hence ν is a K-equivalence. □

3. K-theory of free wreath products

We are now in a position to prove the other main result of this thesis, namely we want to explicitly compute the K-theory of free wreath products of compact quantum groups with S_N^+. As done in [FT24] we need to use Theorem 5.2.1. First let us introduce the notion of K-amenability for compact quantum groups.

Definition 5.3.1 (K-amenable): Let G be a compact quantum group. We say that $\widehat{G}$ is K-amenable if the canonical surjection $\lambda\colon C(G) \to C_r(G)$ from the full to the reduced C^*-algebra is a KK-equivalence.

A useful equivalence is the following, proven by Vergnioux in [Ver04].

Proposition 5.3.2: *Let G be a compact quantum group. Denote by $\varepsilon\colon C(G) \to \mathbb{C}$ the map defined by $\varepsilon(u^\alpha_{ij}) = \delta_{ij}$ for all $\alpha \in \mathrm{Irr}(G)$. We call this map counit or trivial representation of G. Then $\widehat{G}$ is K-amenable if and only if there exists $\alpha \in KK(C_r(G), \mathbb{C})$ such that*

$$[\lambda] \otimes_{C_r(G)} \alpha = [\varepsilon] \in KK(C(G), \mathbb{C}).$$

Now let us quickly mention what the K-theory of S_N^+ is. It was proven by Voigt in [Voi17].

Proposition 5.3.3: *Let $N \geq 4$. Then the quantum permutation group S_N^+ is K-amenable and*

$$K_0(C(S_N^+)) \cong \mathbb{Z}^{(N-1)^2+1} \text{ and } K_1(C(S_N^+)) \cong \mathbb{Z}.$$

Generators for the K_0-group are given by the projections $[1], [u_{ij}] \in K_0(C(S_N^+))$ for $1 \leq i, j \leq N-1$.

Remark 5.3.4: Note that for $N \leq 3$, we have $C_\bullet(S_N^+) = C(S_N) = \mathbb{C}^{N!}$. Thus $K_0(C(S_N)) = \mathbb{Z}^{N!}$ and $K_1(C(S_N)) = 0$.

Theorem 5.3.5: *For any compact quantum group G and integer $N \in \mathbb{N}$ we have,*

$$\begin{aligned} K_0(C_\bullet(G \wr_* S_N^+)) &\cong K_0(C_\bullet(G)) \otimes \mathbb{Z}^{N^2} \oplus K_0(C_\bullet(S_N^+))/\mathbb{Z}^{N^2} \\ &\cong \begin{cases} K_0(C_\bullet(G))^{\oplus N^2}/\mathbb{Z}^{2N-2} & \text{if } N \neq 3 \\ K_0(C_\bullet(G))^{\oplus N^2}/\mathbb{Z}^3 & \text{if } N = 3 \end{cases}, \\ K_1(C_\bullet(G \wr_* S_N^+)) &\cong K_1(C_\bullet(G))^{\oplus N^2} \oplus K_1(C_\bullet(S_N^+)) \\ &\cong \begin{cases} K_1(C_\bullet(G))^{\oplus N^2} \oplus \mathbb{Z} & \text{if } N \geq 4 \\ K_1(C_\bullet(G))^{\oplus N^2} & \text{if } N \leq 3 \end{cases}, \end{aligned}$$

where $C_\bullet(G)$ denotes either the reduced or full C^*-algebra.

Proof. Consider the graph of C^*-algebras we constructed in Section 3 of Chapter 3. Recall that we can equip both graphs of C^*-algebras with conditional expectations.

We need to use the first 6-exact sequence of Theorem 5.2.1, where we choose $C = \mathbb{C}$. We denote S for the map $\sum s_v^* - r_v^*$. Moreover we assume H to be trivial. We obtain by definition of K_0 and K_1 the following diagram

$$\begin{array}{ccccc} \mathbb{Z}^{N^2} & \xrightarrow{S} & \left(K_0(C_\bullet(G)) \otimes \mathbb{Z}^{N^2}\right) \oplus K_0(C_\bullet(S_N^+)) & \longrightarrow & K_0(C_\bullet(G \wr_* S_N^+)) \\ \uparrow & & & & \downarrow \\ K_1(C_\bullet(G \wr_* S_N^+)) & \longleftarrow & \left(K_1(C_\bullet(G)) \otimes \mathbb{Z}^{N^2}\right) \oplus K_1(C_\bullet(S_N^+)) & \xleftarrow[S]{} & \mathbb{Z}^{N^2}. \end{array}$$

Let $\{[e_j] \mid 1 \leq j \leq N\}$ be a basis of $K_0(\mathbb{C}^N) = \mathbb{Z}^N$, where (e_j) is the canonical basis of $\mathbb{C}^N$. Clearly $\{[1 \otimes e_j] \mid 1 \leq j \leq N\} \subseteq K_0(C_\bullet(G) \otimes \mathbb{C}^N)$ is linearly independent and hence for all $1 \leq j \leq N$ also $\{[1 \otimes e_j] - [u_{ji}]\} \subseteq K_0(C(G) \otimes \mathbb{C}^N) \oplus K_0(C_\bullet(S_N^+))$ is also linearly independent. By this clearly the map

$$s_v^* - r_v^* \colon \mathbb{Z}^N \to \left(K_0(C_\bullet(G)) \otimes \mathbb{Z}^N\right) \oplus K_0(C_\bullet(S_N^+)),\ [e_j] \mapsto [1 \otimes e_j] - [u_{ji}]$$

is injective, therefore the map

$$S \colon (\mathbb{Z}^N)^{\oplus N} \to \left(K_0(C_\bullet(G)) \otimes \mathbb{Z}^{N^2}\right) \oplus K_0(C_\bullet(S_N^+)),\ [e_{ij}] \mapsto [1 \otimes e_{ij}] - [u_{ji}],$$

where $[e_{ij}]$ denotes the class of the element $[e_j]$ in the i-th copy of $(\mathbb{Z}^N)^{\oplus N}$, since again the elements of the form $[1 \otimes e_{ij}]$ for $1 \leq i, j \leq N$ are linearly independent. The same holds for the elements of the form $[1 \otimes e_{ij}] - [u_{ji}]$.

Since the image of the second upper arrow is equal to the kernel of the right vertical arrow, and the image of the right vertical arrow is the same as the kernel as the first lower arrow by exactness, we obtain that

$$\left(K_0(C_\bullet(G)) \otimes \mathbb{Z}^{N^2}\right) \oplus K_0(C_\bullet(S_N^+)) \to K_0(C_\bullet(G \wr_* S_N^+))$$

is surjective and by using again exactness we get the isomorphism

$$\left(K_0(C_\bullet(G)) \otimes \mathbb{Z}^{N^2}\right) \oplus K_0(C_\bullet(S_N^+))/\operatorname{Im}(S) \cong K_0(C_\bullet(G \wr_* S_N^+)).$$

By identifying $[1 \otimes e_{ij}]$ with $[u_{ij}]$, we obtain $\operatorname{Im}(S) \cong \mathbb{Z}^{N^2}$, and therefore the statement about K_0 follows directly by Proposition 5.3.3 and Remark 5.3.4.

For the K_1-isomorphism note that S has trivial image by definition, and via an analogue statement as above one shows surjectivity of the map

$$\left(K_1(C_\bullet(G)) \otimes \mathbb{Z}^{N^2}\right) \oplus K_1(C_\bullet(S_N^+)) \to K_1(C_\bullet(G \wr_* S_N^+)).$$

The kernel is then also trivial since the image of S is trivial, and by this the isomorphism. □

Corollary 5.3.6: *The quantum hyperoctahedral group H_N^+ is K-amenable and*

$$K_0(C(H_N^+)) \cong \begin{cases} \mathbb{Z}^{N^2+N!}, & \text{for } N = 1,2,3, \\ \mathbb{Z}^{2N^2-2N+2}, & \text{for } N \geq 4, \end{cases}$$

$$K_1(C(H_N^+)) \cong \begin{cases} 0, & \text{for } N = 1,2,3 \\ \mathbb{Z}, & \text{for } N \geq 4 \end{cases}.$$

Proof. By Proposition 3.5.14, we know that $H_N^+ \cong \widehat{\mathbb{Z}/2\mathbb{Z}} \wr_* S_N^+$. Moreover by Lemma 3.5.13 we have $\widehat{\mathbb{Z}/2\mathbb{Z}} \cong C^*(\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{C}^2$ and

$$K_0(\mathbb{C}^2) = \mathbb{Z}^2,\ K_1(\mathbb{C}^2) = \{0\}.$$

By this the statement follows obviously. □

Corollary 5.3.7: *Let G be a compact quantum group, then $\widehat{G \wr_* S_N^+}$ is K-amenable if and only if $\widehat{G}$ is K-amenable.*

Proof. This follows directly from Theorem 5.3.5. □

More generally we can prove the same for the amalgamated free wreath product. Recall the notations of Section 3 of Chapter 4.

Proposition 5.3.8: *Let G be a compact quantum group, and H be a dual quantum subgroup. Then $\widehat{G}$ is K-amenable if and only if $\widehat{G \wr_{*,H} S_N^+}$ is K-amenable.*

Proof. Assume that $\widehat{G \wr_{*,H} S_N^+}$ is K-amenable, i.e. there exists $\alpha \in KK(\mathcal{A}_r, \mathbb{C})$ such that $[\lambda] \otimes_{\mathcal{A}_r} \alpha = [\varepsilon]$, where $\varepsilon \colon \mathcal{A} \to \mathbb{C}$ is the counit of the free wreath product, by Proposition 5.3.2. We show that $\widehat{G^{*_H N}}$ is K-amenable, since then also

G is K-amenable. Denote by $\pi\colon C(G^{*_H N}) \to \mathcal{A}$ the canonical inclusion. By taking restrictions and using Proposition 4.3.4, we obtain a unital *-homomorphism $\pi_r\colon C_r(G^{*_H N}) \to \mathcal{A}_r$ such that $\pi_r\colon \lambda' = \lambda \circ \pi$, where $\lambda'\colon C(G^{*_H N}) \to C_r(G^{*_H N})$ is the canonical surjection. By defining $\beta := [\pi_r] \otimes_{\mathcal{A}_r} \alpha \in KK(C_r(G^{*_H N}), \mathbb{C})$, we have by functoriality of the Kasparov product

$$\begin{aligned}[\lambda'] \otimes \beta &= [\lambda'] \otimes [\pi_r] \otimes \alpha = [\pi_r \circ \lambda'] \otimes \alpha \\ &= [\lambda \circ \pi] \otimes \alpha = [\pi] \otimes [\lambda] \otimes [\alpha] \\ &= [\pi] \otimes [\varepsilon] = [\varepsilon \circ \pi] = [\varepsilon'],\end{aligned}$$

where $\varepsilon'\colon C(G^{*_H N}) \to \mathbb{C}$ is the counit. Hence by Proposition 5.3.2 we get that $\widehat{G^{*_H N}}$ is K-amenable. Thus $\widehat{G}$ is K-amenable.

For the other direction, let $\widehat{G}$ be K-amenable, then also $\widehat{H}$ as dual quantum subgroup is K-amenable. By Proposition 5.3.3 we know that $\widehat{S_N^+}$ is K-amenable. Since we equipped the graph of C^*-algebras for the free wreath product for the full and reduced case with conditional expectation, we are in the case of Corollary 5.2.5. One then can apply [FF14, Theorem 5.1] to prove that $\widehat{G \wr_{*,H} S_N^+}$ is K-amenable, since our graph is finite. This proves the equivalence. □

4. Questions

To end this bachelor thesis, we would like to address some questions that might arise after reading.

Question 5.4.1 (Free wreath product): For a compact quantum group G we did define the free wreath product $G \wr_* S_N^+$ with S_N^+. In a natural way one can ask if one can generalise this somehow.

Indeed one can see S_N^+ as the *quantum automorphism group*, [Wan98], of $\mathbb{C}^N$ with the trace τ corresponding to the uniform probability measure on N points, this defines us a so called *δ-form*. We then write $S_N^+ = \mathrm{Qut}(\mathbb{C}^N, \tau)$. So we might think about a generalisation, where we take another quantum automorphism group instead of S_N^+, and it could be interesting to try to compute the K-theory of this free wreath product with another quantum automorphism group. See for instance [FP16] for the construction of the free wreath product with a quantum automorphism group.

Another question can arise from Theorem 5.3.5.

Question 5.4.2 (Amalgamation): Looking at the proof of this mentioned theorem, we see that we assumed the dual quantum subgroup to be trivial. Thus it might be interesting to compute the K-theory for a free wreath product with non-trivial amalgamation, since in [FT24] they have at most considered special cases.

A question in a more general context could be the following.

Question 5.4.3 (S_N^+ and H_N^+): How we can see, the computation of the K-theory of H_N^+ but also the K-theory of S_N^+ in [Voi17] is quite complex and one needs a lot of different structure and theory. However both of them are easily described. Therefore it might be interesting to search "easier" ways to compute the K-theory of them, as this could possibly also show other structural properties of S_N^+ and H_N^+.

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