Exercises 2

1) Define an unbounded operator T on $L^2([0,1])$ with domain $D(T) = C([0,1])$ (continuous functions) by

$$Tf = f(0)1,$$

where 1 is the constant function, $1(t) = 1$. Determine the adjoint T^*. (Last time we convinced ourselves that T is not closable, so the domain of T^* should better not be dense.)

2) Let T be the derivative operator from Example 4.7, which was defined on

$$D(T) = \{f \in L^2([0,1]) \mid f \text{ continuously differentiable}, f(0) = 0 = f(1)\}.$$

Show that

$$\{f' \mid f \in D(T)\}^\perp = \mathbb{C}1.$$

3) Consider the derivative operator on $\mathcal{H} = L^2(\mathbb{R})$,

$$D(T) := \{f \in \mathcal{H} \mid f \text{ is AC}, f' \in \mathcal{H}\}, \qquad Tf = if'.$$

Show that T is symmetric. Calculate the adjoint of T. What are the defect indices? How is the situation with selfadjoint extensions?

4) Consider the derivative operator on $\mathcal{H} = L^2([0,\infty))$,

$$D(T) := \{f \in \mathcal{H} \mid f \text{ is AC}, f' \in \mathcal{H}, f(0) = 0\}, \qquad Tf = if'.$$

Show that T is symmetric. Calculate the adjoint of T. What are the defect indices? How is the situation with selfadjoint extensions?

It might be good to know the following properties of absolutely continuous (AC) functions, even if you don't remember their definition. (There will be a video on this, but not right now.)

- **Fundamental Theorem of Calculus (in its most general form)**

  1) Let $f \in L^1[a,b]$ and put

  $$g(t) := \int_a^t f(s)ds.$$

  Then g is AC and $g'(t) = f(t)$ almost everywhere.

  2) Let $g : [a,b] \to \mathbb{C}$ be AC. Then g is differentiable almost everywhere, $g' \in L^1[a,b]$ and

  $$g(t) = g(a) + \int_a^t g'(s)ds.$$

- The product of two AC functions is also AC. (This allows to do partial integration for AC functions.)