SAARLAND UNIVERSITY
Faculty of Mathematics and Computer Science
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Assignment for the lecture Functional Analysis
Winter term 2022/23

Sheet 12
Due on Mon 6.2.2023

Exercise 1 (30 Points)**.** On Sheet 9, Exercise 1 we showed that $l_1(\mathbb{Z})$ endowed with the convolution $*$ as multiplication is a Banach algebra and on Sheet 10, Exercise 1 that every complex homomorphism of $l_1(\mathbb{Z})$ is of the form

$$\varphi_z \colon l_1(\mathbb{Z}) \to \mathbb{C}, \quad (a_n)_{n\in\mathbb{Z}} \mapsto \sum_{n\in\mathbb{Z}} a_n z^n,$$

for a $z \in \mathbb{T} = \{z \in \mathbb{C} \colon |z| = 1\}$, i.e. we can identify $\Sigma(l_1(\mathbb{Z}))$ and $\mathbb{T}$ as sets. Hence the Gelfand transform is exactly the Fourier transform: For $x = (a_n)_{n\in\mathbb{Z}}$ we have

$$\hat{x}(z) \cong \hat{x}(\varphi_z) = \varphi_z(x) = \sum_{n\in\mathbb{Z}} a_n z^n$$

and clearly $\widehat{x * y}(z) = \hat{x}(z) \cdot \hat{y}(z)$ for any $x, y \in l_1(\mathbb{Z})$.

a) Show that $\Sigma(l_1(\mathbb{Z}))$ and $\mathbb{T}$ are isomorphic as topological spaces.

b) Let $x \in A := l_1(\mathbb{Z})$, then $\hat{x}$ is a continuous function on $\mathbb{T}$. Actually, $\hat{A} \subset C(\mathbb{T})$ is a proper subalgebra (note that A cannot be made into a C^*-Algebra). Show that

$$\hat{A} = \{f \in C(\mathbb{T}) \colon f \text{ has an absolutely convergent Fourier series}\},$$

where we say f *has an absolutely convergent Fourier series* if and only if $f(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^n$ with $\sum_{n\in\mathbb{Z}} |\gamma_n| < \infty$.
Give an example of a function $f \in C(\mathbb{T})$ that is not in $\hat{A}$.

c) Show that the Gelfand representation $\hat{\cdot} \colon A \to C(\mathbb{T})$ is injective but not isometric.

d) Prove the *Theorem of Wiener*: If $f \in C(\mathbb{T})$ has an absolutely convergent Fourier series and $f(z) \neq 0$ for all $z \in \mathbb{T}$, then $\frac{1}{f}$ also has an absolutely convergent Fourier series.

Exercise 2 (10 Points)**.** Let A be a C^*-algebra and let $a \in A$ be a normal element. Consider a continuous function on the spectrum of a, $f \in C(\sigma(a))$. Show that $\sigma(f(a)) = f(\sigma(a))$, where $f(\sigma(a)) = \{f(\lambda) \colon \lambda \in \sigma(a)\}$.

Exercise 3 (10 Points)**.** Let $\Phi \colon A \to B$ be a $*$-homomorphism of unital C^*-algebras. Show that Φ is continuous with $\|\Phi\| \leq 1$.
(*Hint:* Show that $\sigma(\Phi(x^*x)) \subset \sigma(x^*x)$)