Exercises for
Theoretical physics V

SoSe 2025 **Sheet 2** **15.04.2025**

Exercise 1 *Conservation equation of quantum probability*

Let us consider a system described by the position- and time-dependent wavefunction $\psi(\boldsymbol{r}, t)$. The behaviour of the wavefunction is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{r}, t) + V(\boldsymbol{r})\psi(\boldsymbol{r}, t)\,. \tag{1}$$

a) Defining the quantum probability density $\rho(\boldsymbol{r}, t) = |\psi(\boldsymbol{r}, t)|^2$, determine the equation of motion of ρ.

(2 points)

b) Show that it can be expressed in terms of a conservation equation

$$\frac{\partial}{\partial t}\rho(\boldsymbol{r}, t) + \nabla\cdot\boldsymbol{j}(\boldsymbol{r}, t) = 0. \tag{2}$$

Give the explicit form of the current of probability $\boldsymbol{j}$.

Hint: One shall use that $\nabla\cdot(f\nabla g) = (\nabla f)\cdot(\nabla g) + f\nabla^2 g$.

(2 points)

Exercise 2 *Pauli matrices and their properties*

The Pauli matrices are defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,,\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{3}$$

a) Show that the Pauli matrices fulfill the following relations

$$\sigma_j\sigma_k = \delta_{jk}\mathbf{1}_2 + i\epsilon_{jkl}\sigma_l \quad\text{and}\quad \sigma_x\sigma_y\sigma_z = i\mathbf{1}_2, \tag{4}$$

where $j, k, l \in \{x, y, z\}$, δ_{jk} stands for the Kronecker delta, and ϵ_{jkl} is the Levi-Civita tensor, such that $\epsilon_{xyz} = \epsilon_{yzx} = \epsilon_{zxy} = 1$, $\epsilon_{yxz} = \epsilon_{zyx} = \epsilon_{xzy} = -1$, otherwise $\epsilon_{jkl} = 0$.

(2 points)

b) Prove that the Pauli matrices follow the commutation and anti-commutation relations:

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l \quad\text{and}\quad \{\sigma_j, \sigma_k\} = 2\delta_{jk}\mathbf{1}_2, \tag{5}$$

where the commutators and anticommutators of two operators A and B are defined as $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$.

(2 points)

c) Show that the vector of Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ fulfill the following relation

$$(\boldsymbol{a} \cdot \boldsymbol{\sigma})(\boldsymbol{b} \cdot \boldsymbol{\sigma}) = \boldsymbol{a} \cdot \boldsymbol{b} + i(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{\sigma}. \tag{6}$$

Use this equality on a normalized vector $\boldsymbol{n} \in \mathbb{R}^3$ to show that

$$(\boldsymbol{n} \cdot \boldsymbol{\sigma})^2 = 1. \tag{7}$$

(2 points)

Exercise 3 *Time ordering*

We consider a system whose dynamics is described by the Schrödinger equation

$$\mathrm{i}\hbar \frac{\partial}{\partial t} \psi(\boldsymbol{r}, t) = H(t)\psi(\boldsymbol{r}, t)\,, \tag{8}$$

with an explicit time-dependent Hamiltonian $H(t)$. The formal solution reads $\psi(\boldsymbol{r}, t) = U(t, 0)\psi(\boldsymbol{r}, t = 0)$, with the initial state $\psi(\boldsymbol{r}, t = 0)$ and the time-evolution operator

$$U(t, 0) = \mathcal{T}\!: \exp\left(\frac{1}{\mathrm{i}\hbar} \int_0^t \mathrm{d}t' H(t')\right)\,. \tag{9}$$

Here, $\mathcal{T}$: denotes the time-ordering operator, whose action on two time-dependent operators $A(t)$ and $B(t)$ is

$$\mathcal{T}\!:(A(t_1)B(t_2)) = \begin{cases} A(t_1)B(t_2), & t_1 > t_2 \\ B(t_2)A(t_1), & t_1 < t_2\,. \end{cases} \tag{10}$$

Show that the definition (9), to second order, corresponds to

$$U(t, 0) = \mathbf{1} + \frac{1}{\mathrm{i}\hbar} \int_0^t H(t')\mathrm{d}t' + \frac{1}{(\mathrm{i}\hbar)^2} \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' H(t')H(t'') + (\text{higher orders})\,. \tag{11}$$

Hint: Start with the definition

$$\exp(O) = \sum_{n=0}^{\infty} \frac{1}{n!} O^n\,, \quad \text{for} \quad O = \frac{1}{\mathrm{i}\hbar} \int_0^t \mathrm{d}t' H(t')\,,$$

and apply the time-ordering operator, accounting for its definition (10):

$$\mathcal{T}\!: \exp(O) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{T}\!:(O^n)\,.$$

(2 points)