

Exercises for Theoretical physics V

SoSe 2025

Sheet 1

8.04.2025

Exercise 1 *Quantum harmonic oscillator*

Let us consider a particle of mass m in one dimension. We define the operator position \hat{x} and impulsion \hat{p} such that they obey the commutation relation $[\hat{x}, \hat{p}] = i\hbar\mathbf{1}$. The particle is trapped in a harmonic potential such that it is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \quad (1)$$

where $\omega \in \mathbb{R}$.

- a) Let us define the annihilation and creation operators $\hat{a} = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{l} + i\frac{l\hat{p}}{\hbar}\right)$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{\hat{x}}{l} - i\frac{l\hat{p}}{\hbar}\right)$, where $l = \sqrt{\frac{\hbar}{m\omega}}$. Show that $[\hat{a}, \hat{a}^\dagger] = \mathbf{1}$ and that $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}\mathbf{1})$

(1 point)

- b) For $\hat{N} = \hat{a}^\dagger\hat{a}$, show then that $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$.

(1 point)

- c) Let us consider the eigenvectors $|n\rangle$ of the operator \hat{N} and \hat{H} , such that $\hat{N}|n\rangle = n|n\rangle$, where $n \in \mathbb{N}$. Show that if the ground state $|0\rangle$ is not degenerate, then so are the other states $|n\rangle$.

(1 point)

- d) Let us consider the state $|\psi_0\rangle = \sum_n c_n|n\rangle$, where $c_n = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}}$ and $\alpha \in \mathbb{C}$. Show that $\langle\psi_0|\psi_0\rangle = 1$. Determine the expectation values $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{x}^2\rangle$ and $\langle\hat{p}^2\rangle$.

(2 points)

- e) Determine the expression of the evolved state $|\psi(t)\rangle = \hat{U}(t)|\psi_0\rangle$. What form takes then $\langle\hat{x}(t)\rangle$ and $\langle\hat{p}(t)\rangle$?

(2 points)

Exercise 2 *Relativistic notations*

- a) Let us define the four-momentum vector $P^\mu = mu^\mu = (E/c, p_x, p_y, p_z)$. Show that its pseudo-norm is constant and deduce the relation

$$E^2 = p^2c^2 + m^2c^4. \quad (2)$$

Assuming that the mass term is dominant ($m^2c^4 \gg p^2c^2$), determine the form of the energy E .

(2 points)

- b) The Maxwell tensor $F_{\mu\nu}$ is defined from the four-potential $A^\mu = (\phi, A_x, A_y, A_z)$ (in Gauss units) as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Show that in the Lorenz gauge ($\partial_\mu A^\mu = 0$), the equation of motion for the electric potential ϕ and the vector potential \mathbf{A} take the compact form

$$\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} j^\nu, \quad (3)$$

where the four-current is defined as $j^\nu = (c\rho, j_x, j_y, j_z)$.

Hint: In Gaussian units, the Maxwell equations take the form

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \quad (4a)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4b)$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \mathbf{B}, \quad (4c)$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \left(4\pi \mathbf{j} + \frac{\partial}{\partial t} \mathbf{E} \right). \quad (4d)$$

(2 points)