Theoretical physics V
Sheet 14

SoSe 2024 **Due for the 25.07.2024**

Exercise 34 *Energy conservation and time-translational invariance*

Let us consider the Lagrangian density associated to a real-valued bosonic field

$$\mathcal{L} = \frac{1}{2}(\partial^{\nu}\psi)(\partial_{\nu}\psi) - V(\phi^2), \tag{1}$$

where V is an arbitrary continuous function. Show that for such a Lagrangian density, the conservation of the energy implies the time translation symmetry of the system.

(2 points)

Exercise 35 *Quantized Dirac field*

Let us recall that the quantized Dirac field can be written in terms of fermionic creation and annihilation operators $a_{\boldsymbol{p},s}$ and $b_{\boldsymbol{p},s}$, such that for a periodic box-like space of volume Ω

$$\hat{\psi}(\boldsymbol{r},t) = \sum_{\boldsymbol{p},s}\sqrt{\frac{mc^2}{\Omega E_{\boldsymbol{p}}}}\left[\hat{a}_{\boldsymbol{p},s}u(\boldsymbol{p},s)e^{i(\boldsymbol{p}\cdot\boldsymbol{r}-E_{\boldsymbol{p}}t)/\hbar} + \hat{b}^{\dagger}_{\boldsymbol{p},s}v(\boldsymbol{p},s)e^{-i(\boldsymbol{p}\cdot\boldsymbol{r}-E_{\boldsymbol{p}}t)/\hbar}\right], \tag{2}$$

with $u^{\dagger}(\boldsymbol{p},s)u(\boldsymbol{p},s') = v^{\dagger}(\boldsymbol{p},s)v(\boldsymbol{p},s') = E_{\boldsymbol{p}}\delta_{ss'}/mc^2$. In other words, the operator $\hat{a}_{\boldsymbol{p},s}$ destroys a particle with wavefunction $u(\boldsymbol{p},s)$, while $\hat{b}_{\boldsymbol{p},s}$ annihilates an antiparticle with wavefunction $v^{\dagger}(\boldsymbol{p},s)$.

a) Using the following definitions,

$$\hat{H} = \int \mathrm{d}^3\boldsymbol{r}\hat{\psi}^{\dagger}(\boldsymbol{r},t)(-i\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta mc^2)\hat{\psi}(\boldsymbol{r},t), \tag{3}$$

$$\hat{\boldsymbol{P}} = -i\hbar\int \mathrm{d}^3\boldsymbol{r}\hat{\psi}^{\dagger}(\boldsymbol{r},t)\boldsymbol{\nabla}\hat{\psi}(\boldsymbol{r},t), \tag{4}$$

$$\hat{Q} = -e\int \mathrm{d}^3\boldsymbol{r}\hat{\bar{\psi}}(\boldsymbol{r},t)\gamma^0\hat{\psi}(\boldsymbol{r},t), \tag{5}$$

determine the form of the Hamiltonian $\hat{H}$, the momentum operator $\hat{\boldsymbol{P}}$ and the charge $\hat{Q}$ in terms of operator $\hat{a}_{\boldsymbol{p},s}$ and $\hat{b}_{\boldsymbol{p},s}$.

(3 points)

b) Considering the two-particle state $|\Psi\rangle = \hat{a}^{\dagger}_{\boldsymbol{p},s}\hat{a}^{\dagger}_{\boldsymbol{p}',s'}|0\rangle$, show that such a state is antisymmetric and determine the expectation values of $\hat{H}$, $\hat{\boldsymbol{P}}$ and $\hat{Q}$.

(1 point)

c) For a two-anti-particle state $|\Psi\rangle = \hat{b}^{\dagger}_{\boldsymbol{p},s}\hat{b}^{\dagger}_{\boldsymbol{p}',s'}|0\rangle$, perform the same calculations as in the two-particle state case.

(1 point)