# Theoretical physics V
## Sheet 13

**SoSe 2024** **Due for the 18.07.2024**

## Exercise 32 *The complex scalar field*

Let us consider the Lagrangian density associated to a complex-valued Klein-Gordon field $\psi$:

$$\mathcal{L} = (\partial^\nu \psi)(\partial_\nu \psi)^* - \mu^2 |\psi|^2, \tag{1}$$

where $\mu = mc/\hbar$.

a) Show that the Lagrangian density is invariant under the transformation $\psi \to \psi' = e^{i\lambda}\psi$.

**Hint:** You shall consider an infinitesimal transformation for $\lambda \ll 1$ and determine the variation of $\delta\mathcal{L}$.

*(1 point)*

b) Using the prescription of minimal coupling, we define the gauge covariant derivative as

$$-i\hbar\partial_\mu \to -i\hbar D_\mu = -i\hbar\partial_\mu - eA_\mu, \tag{2}$$

derive the field equation for $\psi$ interacting with $A_\mu$. Show that if $\psi$ is a solution he equation of motion for $A_\mu = (A_0, 0, 0, 0)$, then $\psi^*$ is a solution when $A_0$ is replaced by $-A_0$.

*(2 points)*

## Exercise 33 *The Jordan-Wigner transformation*

Let us consider a chain of evenly spaced particles with spins $S = 1/2$. In a ferromagnetic system, the spins of these particles interact such as to align, therefore leading to an emergent long-range order. Several types of Hamiltonian describe such a behavior, we will however focus here on the case of the so called quantum $XY$-model:

$$\hat{H} = -J \sum_{i=1}^{N} \left( \hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y \right), \tag{3}$$

where we assume periodic boundary conditions, namely $\hat{S}_{N+1}^{x,y} = \hat{S}_1^{x,y}$.

a) Defining the raising and lowering spin operators as $\hat{S}_j^\pm = (\hat{S}_j^x \pm i\hat{S}_j^y)$, show that the spin commutation relation $[\hat{S}_i^\alpha, \hat{S}_j^\beta] = i\epsilon^{\alpha\beta\gamma}\delta_{ij}\hat{S}_j^\gamma$ leads to the relations

$$[\hat{S}_i^z, \hat{S}_j^\pm] = \pm\delta_{ij}\hat{S}_j^\pm \quad ; \quad [\hat{S}_i^+, \hat{S}_j^-] = 2\delta_{ij}\hat{S}_j^z. \tag{4}$$

*(1 point)*

b) In a one-dimensional system, a way to treat the problem of spin chains (for $S = 1/2$) consists in associating the orientation of a spin to the presence or abscence of a fermion, in other words $|\uparrow\rangle_j \equiv |1\rangle_j = \hat{f}_j^\dagger |0\rangle_j$ while $|\downarrow\rangle_j \equiv |0\rangle_j$. Note that the Pauli exclusion principle encodes that only two spin orientations are possible. There is however a discrepancy between the algebra describing these two types of object: while spin operators commute, fermionic operators anticommute. In order to solve this problem, we introduce the so-called *Jordan-Wigner transformation*, such that

$$\hat{S}_i^+ = \hat{f}_i^\dagger e^{i\pi \sum_{k<i} \hat{n}_k}\,, \quad \hat{S}_i^- = e^{-i\pi \sum_{k<i} \hat{n}_k} \hat{f}_i\,, \quad \hat{S}_i^z = \hat{f}_i^\dagger \hat{f}_i - \frac{1}{2}\,. \tag{5}$$

Show the anticommutation relation $\{e^{i\pi \hat{n}_i}, \hat{f}_i\} = 0$, then deduce that the Jordan-Wigner transformation preserves the spin commutation relations.

*(3 points)*

c) Using the Jordan-Wigner transformation, show that the Hamiltonian of the $XY$-model takes the simple form

$$\hat{H} = -\frac{J}{2} \sum_{i=1}^{N} \left( \hat{f}_{i+1}^\dagger \hat{f}_i + \hat{f}_i^\dagger \hat{f}_{i+1} \right). \tag{6}$$

Then, via a Fourier transform $\hat{f}_j = 1/\sqrt{N} \sum_k \hat{f}_k e^{ikj}$, show that the Hamiltonian reads

$$\hat{H} = \sum_k \omega_k \hat{f}_k^\dagger \hat{f}_k. \tag{7}$$

Give the explicit form of $\omega_k$.

**Hint:** We remind that

$$\sum_j e^{i(k-k')j} = N\delta_{kk'}.$$

*(2 points)*

d) Based on the interpretation given for the negative-energy solutions to the Dirac equation, comment on the form of $\omega_k$, the nature of the ground state of the $XY$-model and its excitations.

*(1 point)*