

Theoretical physics V

Sheet 12

SoSe 2024

Due for the 11.07.2024

Exercise 30 *Bosonic and fermionic fields operators*

a) The basic commutation relation for boson annihilation and creation operator is

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad [\hat{a}, \hat{a}] = 0, \quad (1)$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. From this definition show that eigenstates $|n\rangle$ of $\hat{a}^\dagger\hat{a}$ have the properties

$$\begin{aligned}\hat{a}^\dagger\hat{a}|n\rangle &= n|n\rangle, \quad (n \in \mathbb{N}), \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle, \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle.\end{aligned}$$

(2 points)

b) Fermionic annihilation and creation operators are defined by an anticommutation relation

$$\{\hat{a}, \hat{a}^\dagger\} = 1, \quad \{\hat{a}, \hat{a}\} = 0, \quad (2)$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. Show that

$$\begin{aligned}\hat{a}^\dagger\hat{a}|n\rangle &= n|n\rangle, \quad (n = 0, 1), \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle, \\ \hat{a}^\dagger|n\rangle &= \sqrt{1-n}|n+1\rangle.\end{aligned}$$

(2 points)

Exercise 31 *Second quantization and the Schrödinger equation*

Let us consider the N -body wave function defined as

$$\Psi_E(\mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{1}{\sqrt{N!}} \langle 0 | \hat{\psi}(\mathbf{r}_1) \dots \hat{\psi}(\mathbf{r}_N) | E, N \rangle, \quad (3)$$

where $|E, N\rangle$ is an N -particles energy eigenstate with eigenvalue E of the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m} \int d^3\mathbf{r} \hat{\psi}^\dagger(\mathbf{r}) \nabla^2 \hat{\psi}(\mathbf{r}) + \frac{1}{2} \int d^3\mathbf{r}_1 d^3\mathbf{r}_2 \hat{\psi}^\dagger(\mathbf{r}_1) \hat{\psi}^\dagger(\mathbf{r}_2) v(\mathbf{r}_1, \mathbf{r}_2) \hat{\psi}(\mathbf{r}_2) \hat{\psi}(\mathbf{r}_1). \quad (4)$$

a) Show that the wave function Ψ_E is normalized to unity, namely that

$$\int d^3\mathbf{r}_1 \dots d^3\mathbf{r}_N |\Psi_E(\mathbf{r}_1, \dots, \mathbf{r}_N)|^2 = 1. \quad (5)$$

(1 point)

b) Show also that

$$E\Psi_E(\mathbf{r}_1, \dots, \mathbf{r}_N) = \frac{1}{\sqrt{N!}} \langle 0 | \hat{\psi}(\mathbf{r}_1) \dots \hat{\psi}(\mathbf{r}_N) \hat{H} | E, N \rangle. \quad (6)$$

(1 point)

c) Show that the wave function satisfies the N -particles Schrödinger equation

$$\left[-\sum_{i=1}^N \frac{\hbar^2}{2m} \nabla_i^2 + \sum_{i < j} v(\mathbf{r}_i, \mathbf{r}_j) \right] \Psi_E(\mathbf{r}_1, \dots, \mathbf{r}_N) = E\Psi_E(\mathbf{r}_1, \dots, \mathbf{r}_N). \quad (7)$$

(2 points)