Theoretical physics V
Sheet 10

SoSe 2024 **Due for the 27.06.2024**

Exercise 26 *Jaynes-Cummings model*

Let us approximate an atom by a two-level system consisting of a ground state $|g\rangle$ and an excited state $|e\rangle$. This atom is set in an optical cavity formed by two mirrors, such that light-fied within the cavity oscillates at a single frequency ω. The Hamiltonian of this system reads

$$\hat{H}_0 = \hbar\omega_0 |e\rangle\langle e| + \hbar\omega \hat{a}^\dagger \hat{a}, \tag{1}$$

where $\hat{a}$ is annihilates a quantum of excitationof the light-field. The light-matter interaction in this context is modeled by an interaction Hamiltonian

$$\hat{V} = \hbar g \hat{\sigma}^x \otimes (\hat{a}^\dagger + \hat{a}), \tag{2}$$

where $\hat{\sigma}^x = |g\rangle\langle e| + |e\rangle\langle g|$. This is the celebrated Jaynes-Cummings model. In the following, we will assume that the cavity is resonant with the transition frequency of the atom $\omega = \omega_0$.

a) Determine the matrix element of the interaction term $\hat{V}$ in the basis $\{|g,n\rangle, |e,n\rangle\}$ combining the two levels $\{|g\rangle, |e\rangle\}$ of the atom and the levels $\{|n\rangle\}$ of the harmonic oscillator. Represent graphically the processes induced by the light-matter interaction and identify which one are resonant.

(2 points)

b) Determine up to second order in perturbation theory the energy shift of the ground state $|g,0\rangle$ in the weak-coupling approximation ($g \ll \omega_0$).

Hint: The first- and second-order energy corrections for a state $|\psi\rangle$ perturbed by a term $\hat{V}$ takes the form

$$\delta E^{(1)} = \langle\psi|\hat{V}|\psi\rangle \quad \text{and} \quad \delta E^{(2)} = \lim_{\eta\to 0} \langle\psi|\hat{V} \frac{1}{E_0 - \hat{H}_0 + i\eta} \hat{V}|\psi\rangle,$$

where $\hat{H}_0|\psi\rangle = E_0|\psi\rangle$, H_0 being the unperturbed Hamiltonian. *(1 point)*

c) Assuming that at time $t = 0$, the atom and the cavity are in state $|_p si_0\rangle = |e,0\rangle$, determine the dynamics of the system.

Hint: We shall only consider resonant processes.

(1 point)

Exercise 27 *Classical limit of electrodynamics*

Let us consider a polarized monochromatic electromagnetic wave of frequency ω with a wavevector $\boldsymbol{k}$. The vector potential associated to the wave is described by the operator

$$\hat{\boldsymbol{A}}(\boldsymbol{r}, t) = \sqrt{\frac{c^2\hbar}{2\omega V}}\left[\hat{a}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right]\boldsymbol{e}, \tag{3}$$

where $\boldsymbol{e}$ is real-valued.

a) Determine the form of the corresponding electric field operator $\hat{\boldsymbol{E}}(\boldsymbol{r}, t)$ and compute the expectation value $\langle n|\hat{\boldsymbol{E}}(\boldsymbol{r}, t)|n\rangle$ for any Fock state $|n\rangle$.

(1 point)

b) Coherent states of an harmonic oscillator are defined as eingenstates of the annihilation operator: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where $\alpha \in \mathbb{C}$. Defining $\alpha = \sqrt{\overline{n}}e^{i\phi}$, determine the expectation value of the electric field $\langle\alpha|\hat{\boldsymbol{E}}(\boldsymbol{r}, t)|\alpha\rangle$. Give an interpretation of the number $\overline{n}$.

(2 points)

c) Evaluate the variance of the amplitude of the electric field $\Delta E = \sqrt{\langle\alpha|\hat{\boldsymbol{E}}^2|\alpha\rangle - \langle\alpha|\hat{\boldsymbol{E}}|\alpha\rangle^2}$, then compare it to the amplitude of the electric field. In which limit do we recover the behavior of a classical system?

(2 points)

Exercise 28 *Photoelectric effect*

Let us consider a state $|g\rangle$ at energy $\hbar\omega_g = 0$ that couples to a continuum of states $|j\rangle$ by exchanging excitations with a quantum harmonic oscillator. In that case, the total Hamiltonian reads $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$, where

$$\hat{H}_0 = \sum_{j=0}^{+\infty} \hbar(\omega_0 + \delta j)|j\rangle\langle j| + \hbar\Omega\hat{a}^\dagger\hat{a}\,, \tag{4}$$

$$\hat{H}_{\text{int}} = \hbar\lambda\sum_{j=0}^{+\infty}(|j\rangle\langle g| + |g\rangle\langle j|)(\hat{a} + \hat{a}^\dagger). \tag{5}$$

The interaction term $\hat{H}_{\text{int}}$ is treated as a perturbation to $\hat{H}_0$. We note that $\delta > 0$ and that $\Omega > \omega_0$ is the oscillator frequency. The operator $\hat{a}$ ($\hat{a}^\dagger$) annihilates (creates) a quantum of energy $\hbar\Omega$.

a) Determine the non-vanishing matrix elements of $\hat{H}_{\text{int}}$ that couple the initial state $|\psi_0\rangle = |g, 1\rangle$ to the other states of the system.

(1 point)

b) By the means of the Fermi golden rule, determine the system's transition rate Γ out of the initial state $|\psi_0\rangle = |g, 1\rangle$. We will assume that the continuum of states $|j\rangle$ is described by a density of states $\rho(E)$.

(2 points)

c) Determine, up to second-order in perturbation theory, the energy shift of the state $|g, 0\rangle$.

(1 point)