Theoretical physics V
Sheet 9

SoSe 2024 **Due for the 20.06.2024**

Exercise 23 *The Yukawa potential*

Pions ($\pi^\pm$ and π^0) are mesonic particles that are involved in an effective description of the strong nuclear interaction at the level of the nuclei as they bind protons and neutrons together. As they are scalar bosons ($S = 0$), they are described by a real scalar field ϕ whose Lagrangian density reads

$$\mathcal{L} = \frac{\hbar^2}{m}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - \rho\phi\right], \tag{1}$$

where $\mu = \dfrac{mc}{\hbar}$, with m being the mass of a pion and ρ corresponding to a source term.

a) Determine the Hamiltonian density corresponding to the field ϕ.

b) Show that the equation of motion describing the field ϕ takes the form

$$\partial_\mu\partial^\mu\phi + \mu^2\phi = -\rho. \tag{2}$$

(1 point)

c) In the abscence of the source term ($\rho = 0$), solve the Euler-Lagrange equation of the field ϕ. We shall use the definition of the Fourier transform in a d-dimensional space

$$\phi(\boldsymbol{x}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{d}^d\boldsymbol{x}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{\phi}(\boldsymbol{k}), \tag{3}$$

where $\hat{\phi}(\boldsymbol{k})$ are the Fourier component of ϕ for the vector $\boldsymbol{k}$.

Hint: We give the useful identity

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i), \tag{4}$$

where x_i are the roots of the equation $f(x) = 0$.

(2 points)

d) In the case of a static field, determine the form of the Green function associated to the Euler-Lagrange equation Eq. (2). Then, one may see that the pion field ϕ mediates an effective interaction expressed in terms of the so-called Yukawa potential. Comment on its range.

Hint: You shall use that

$$\int_{-\infty}^{+\infty} \frac{k}{k^2 + \mu^2}\sin(kr)\mathrm{d}k = \pi e^{-\mu r}.$$

(3 points)

Exercise 24 *Photons with circular polarization*

Let us consider a monochromatic electromagnetic field with a wavevector $\boldsymbol{k}$ and two possible linear polarisations $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, such that it respect the conditions $\boldsymbol{k}\cdot\boldsymbol{e}_1 = \boldsymbol{k}\cdot\boldsymbol{e}_2 = \boldsymbol{e}_1\cdot\boldsymbol{e}_2 = 0$. The vectors $\boldsymbol{e}_{1,2}$ are unit vectors.
The energy of the quantized electromagnetic field reads

$$\hat{H} = \hbar c|\boldsymbol{k}|(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1), \tag{5}$$

where the operators $\hat{a}_j^\dagger$ act on the vacuum to create a photon at wavelength $\boldsymbol{k}$ and polarization $\boldsymbol{e}_j$. The annhihilation and creation operators obey bosonic commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \tag{6}$$

$$[\hat{a}_i^{(\dagger)}, \hat{a}_j^{(\dagger)}] = 0. \tag{7}$$

a) We shall consider the circular polarization vectors

$$\boldsymbol{e}_\pm = \mp\frac{1}{\sqrt{2}}(\boldsymbol{e}_1 \pm i\boldsymbol{e}_2). \tag{8}$$

Using the definition of the quantized photon field

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{c}{\sqrt{V}}\sqrt{\frac{\hbar}{2\omega}}\sum_j\left[\hat{a}_j(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{e}_j + \hat{a}_j^\dagger(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{e}_j^*\right], \tag{9}$$

show that it can written in terms of operators $\hat{a}_\pm$ and $\hat{a}_\pm^\dagger$ annihilating and creating photons with polarization $\boldsymbol{e}_\pm$. Determine the explicit form of $\hat{a}_\pm$ and show that they obey to bosonic commutation relations.

(2 points)

b) Let us apply to $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ a rotation around the wavevector $\boldsymbol{k}$ by an infinitesimal angle $\delta\theta$. Show that the polarization vectors are changed by a factor $\delta\boldsymbol{e}_\pm = \mp i\delta\theta\boldsymbol{e}_\pm$. What can you deduce about the spin of the photon ?

Hint: We remind that for a two-level system, the rotation operator $\hat{R}(\theta)$ by an angle θ around the axis $\boldsymbol{n}$ reads

$$\hat{R}(\theta) = \exp(i\theta(\boldsymbol{n}\cdot\boldsymbol{\sigma})) = I_2\cos\theta + i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin\theta,$$

where the elements of the vector $\boldsymbol{\sigma}$ are Pauli matrices.

(2 points)

Exercise 25 *Effective cutoff to the electromagnetic field*

Let us consider an electromagnetic field contained within a quantization volume V with periodic boundary conditions. The Hamiltonian of this field reads

$$\hat{H} = \frac{1}{8\pi}\int_V \mathrm{d}^3\boldsymbol{r}(\hat{\boldsymbol{E}}^2(\boldsymbol{r}) + \hat{\boldsymbol{B}}^2(\boldsymbol{r})), \tag{10}$$

with $\boldsymbol{E}(\boldsymbol{r}) = \sum_\lambda(\boldsymbol{E}_\lambda e^{i\boldsymbol{k}_\lambda\cdot\boldsymbol{r}}\hat{a}_\lambda + \text{h.c.})$ and $\boldsymbol{E}_\lambda = \sqrt{\frac{2\pi\hbar\omega_\lambda}{V}}\boldsymbol{\epsilon}_\lambda$. Each mode λ is characterized by vectors $\boldsymbol{k}_\lambda$ and $\boldsymbol{\epsilon}_\lambda$, with $\omega_\lambda = c|\boldsymbol{k}_\lambda|$ and $\boldsymbol{k}_\lambda = \frac{2\pi}{V^{1/3}}(n_x, n_y, n_z)$, $n_j \in \mathbb{Z}$. Using the bosonic commutation relations $[\hat{a}_\lambda, \hat{a}^\dagger_{\lambda'}] = \delta_{\lambda\lambda'}$, the Hamiltonian can be brought into the form

$$\hat{H} = \sum_\lambda \hbar\omega_\lambda\left(\hat{a}^\dagger_\lambda\hat{a}_\lambda + \frac{1}{2}\right). \tag{11}$$

a) We introduce the average electric field $\boldsymbol{E}_{\text{av}}(\boldsymbol{r})$ as

$$\boldsymbol{E}_{\text{av}}(\boldsymbol{r}) = \int_V \mathrm{d}^3\boldsymbol{\rho}\, f(\boldsymbol{\rho})\boldsymbol{E}(\boldsymbol{r}-\boldsymbol{\rho}), \quad \text{with} \quad f(\boldsymbol{\rho}) = f(\rho) = \frac{1}{(\pi\Delta^2)^{3/2}}\exp(-\rho^2/\Delta^2)\,. \tag{12}$$

Using definition (12), determine the value of the expectation value $\langle\text{vac}|\boldsymbol{E}_{\text{av}}(\boldsymbol{r})|\text{vac}\rangle$. Show as well that

$$\langle\text{vac}|\boldsymbol{E}_{\text{av}}(\boldsymbol{r})\cdot\boldsymbol{E}_{\text{av}}(\boldsymbol{r})|\text{vac}\rangle = \sum_\lambda |\boldsymbol{E}_\lambda|^2\overline{f}(\boldsymbol{k}_\lambda), \tag{13}$$

where $\overline{f}$ is a function of f and $\boldsymbol{k}_\lambda$, and $|\text{vac}\rangle$ denotes the vacuum state of the electromagnetic field.

(2 points)

b) Determine the explicit form of $\overline{f}(\boldsymbol{k}_\lambda)$ introduced in Eq. (13) as a function of Δ. Discuss the expectation values determined previously in the limits $\Delta \to 0$ and $\Delta \to +\infty$.

(2 points)

c) Using the dispersion relation $\omega_\lambda = c|\boldsymbol{k}_\lambda|$, rewrite $\overline{f}$ as a function of ω_λ. Show that Δ defines the cutoff frequency ω_c.

(1 point)

d) Using the cutoff function $\overline{f}(\omega_\lambda)$, demonstrate that the zero-point energy

$$E_0 = \langle\text{vac}|\sum_\lambda \overline{f}(\omega_\lambda)\hbar\omega_\lambda(\hat{a}^\dagger_\lambda\hat{a}_\lambda + \frac{1}{2})|\text{vac}\rangle$$

takes the simple form

$$E_0 = V\frac{\hbar}{4\pi^2c^3}\omega_c^4\,. \tag{14}$$

Note that the summation $\sum_\lambda$ is compact notation for $\sum_{\boldsymbol{k}_\lambda}\sum_{\boldsymbol{\epsilon}\perp\boldsymbol{k}_\lambda}$. Determine the asymptotic behavior of E_0 as a function of V and as a function of ω_c. *(2 points)*

Hints: We provide two identities on Gaussian integrals:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}} e^{+\frac{b^2}{4a}}\,, \quad \mathrm{Re}[a] > 0\,,$$

$$\int_{0}^{+\infty} \mathrm{d}x\, x^{2n+1} e^{-x^2/a^2} = \frac{n!}{2} a^{2n+2}\,, \quad \mathrm{Re}[a^2] > 0\,, \quad n \geq 0\,.$$