Theoretical physics V
Sheet 7

SoSe 2024 **Due for the 06.06.2024**

Exercise 19 *Sine-Gordon equation*

Let us consider an infinite chain of identical pendulums of mass m and length ℓ attached to a common axis. Neighbouring pendulums are coupled via torsion springs of stiffness C. Therefore, the angles θ_n accounting for the angular deviation of the n-th pendulum from its equilibrium position is described by the Lagrangian

$$L = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{2} m\ell^2 \dot{\theta}_n^2 - \frac{1}{2} C(\theta_n - \theta_{n-1})^2 - mg\ell(1 - \cos\theta_n) \right] \tag{1}$$

where g is Earth's gravitational acceleration.

a) Assuming that the distance a between two successive pendulums is the smallest length scale of the system, then we can treat θ as a continuous field such that $\theta_n(t) = \theta(x = na, t)$. Show then that the Lagrangian takes the form

$$L = \int_{-\infty}^{+\infty} \mathrm{d}x \left[\frac{1}{2} J \left(\frac{\partial \theta}{\partial t} \right)^2 - \frac{1}{2} K \left(\frac{\partial \theta}{\partial x} \right)^2 - \Omega(1 - \cos\theta) \right]. \tag{2}$$

Specify the expression of the constants J, K and Ω. *(2 points)*

b) Derive the corresponding Euler-Lagrange equation and show that it takes the form of the sine-Gordon equation:

$$\frac{\partial^2 \theta}{\partial t^2} - c_0^2 \frac{\partial^2 \theta}{\partial x^2} + \omega_0^2 \sin\theta = 0, \tag{3}$$

where $c_0^2 = \frac{Ca^2}{m\ell^2}$ and $\omega_0^2 = \frac{g}{\ell}$.

(1 point)

c) Using the change of variable $z = x - vt$, show that the sine-Gordon equation boils down to

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} = \frac{\omega_0^2}{c_0^2 - v^2} \sin\theta, \tag{4}$$

where $v < c_0$ behaves as a velocity.

(1 point)

d) Solutions of the sine-Gordon equation describe a type of waves called solitons. In order to describe a single soliton, we impose boundary conditions such that $\mathrm{d}\theta/\mathrm{d}z = 0$ for $z \to \pm\infty$, as well as $\theta = 0$ for $z \to -\infty$ and $\theta = 2\pi$ for $z \to +\infty$. Show then that

$$\frac{1}{2} \left(\frac{\mathrm{d}\theta}{\mathrm{d}z} \right)^2 = \frac{\omega_0^2}{c_0^2 - v^2} (1 - \cos\theta). \tag{5}$$

(2 points)

e) Using the relation

$$\int \frac{\mathrm{d}\theta}{\sin(\frac{\theta}{2})} = 2\ln\left(\tan\left(\frac{\theta}{4}\right)\right),$$

determine the form of the solution $\theta(x,t)$ of the sine-Gordon equation. Plot its form for a fixed time t.

(3 points)

Exercise 20 *Sinusoidal perturbation and resonance*

Let us consider a quantum system described by a Hamiltonian $\hat{H}_0$ diagonalized by the set of vectors $|\epsilon_n\rangle$. This system is perturbed by an additional time-dependent term such that the Hamiltonian now reads $\hat{H} = \hat{H}_0 + \lambda\hat{V}(t)$, where

$$\hat{V}(t) = \hat{V}_0 \sin(\omega t). \tag{6}$$

a) Determine at first order in perturbation theory the probability amplitude $w_{i\to f}(t)$ for the system to be in state $|f\rangle$ (at energy $E_f = \hbar\omega_f$) at time t, knowing that the system was initially in state $|i\rangle$ (at energy $E_i = \hbar\omega_i$) at time $t = 0$. Express it in terms of the transition frequency $\omega_{fi} = \omega_f - \omega_i$, and the matrix element $V_{fi} = \langle f|\hat{V}_0|i\rangle$.

(1 point)

b) Determine the expression of the transition probability $p_{i\to f}(t;\omega)$, then take the limit when $|\omega - \omega_{fi}| \ll |\omega_{fi}|$. Show then that for a fixed time t, the probability $p_{i\to f}(\omega)$ admits a maximum.

(2 points)

Exercise 21 *Lorentz transformation and space contraction*

Let us consider a fermionic particle of mass m trapped in a volume V such that it is described by the spinor

$$\psi(\boldsymbol{x},t) = \frac{1}{\sqrt{V}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-ip_\mu x^\mu/\hbar}. \tag{7}$$

A Lorentz boost is applied to the system such that in the new reference frame, the particle moves along the z-axis with a rapidity χ, such that $\tanh\chi = \beta = pc/E$. Therefore, the spinor is transformed such that $\psi'(\boldsymbol{x}',t') = S\psi(\boldsymbol{x},t)$, where

$$S = \cosh\frac{\chi}{2}\mathbf{1}_4 + \alpha_z \sinh\frac{\chi}{2}. \tag{8}$$

Compute the density of probability $\rho' = \psi'^{\dagger}\psi'$ and compare it to $\rho = \psi^{\dagger}\psi$ in the reference frame of the particle at rest.

Hint: Use that for a particle moving at a velocity v, its energy can be related to its mass via the relation $E = \gamma mc^2$, where γ is Lorentz factor.

(2 points)