Theoretical physics V
Sheet 6

SoSe 2024 **Due for the 29.05.2024**

Exercise 16 *About the diffraction function*

Let us define the diffraction function as

$$\delta^{(T)}(E_f - E_i) = \frac{1}{2\pi\hbar}\int_{-T/2}^{T/2} \mathrm{d}\tau e^{i(E_f - E_i)\tau/\hbar} = \frac{1}{\pi}\frac{\sin[(E_f - E_i)T/2\hbar]}{E_f - E_i}, \tag{1}$$

where $\lim_{T\to+\infty}\delta^{(T)}(E_f - E_i) = \delta(E_f - E_i)$, with $\delta(x)$ being the Dirac delta function. In the following, we will demonstrate some properties of the diffraction function.

a) Plot the diffraction function $\delta^{(T)}(x)$ as a function of x.

(1 point)

b) Show the following identity

$$\int_{-\infty}^{+\infty} \mathrm{d}E\delta^{(T)}(E - E_i)\delta^{(T)}(E - E_f) = \delta^{(T)}(E_i - E_f), \tag{2}$$

for T finite.

(1 point)

c) Given the form of $\delta^{(T)}(E)$ provided in Eq. (1), deduce that

$$\int_{-\infty}^{+\infty} \mathrm{d}E_f[\delta^{(T)}(E_f - E_i)]^2 = \frac{T}{2\pi\hbar}. \tag{3}$$

Hint: Use that

$$\int_{-\infty}^{+\infty} \mathrm{d}x\frac{\sin^2 x}{x^2} = \pi.$$

(2 points)

Exercise 17 *Ionization of an atom*

Let us consider a simplified model for the ionization of an atom consisting in a ground state $|g\rangle$ at energy $\hbar\omega_0$ and a continuum of ionized states labeled $|I\rangle$ at energy $\hbar(\omega_1 + \omega_I)$, such that $\omega_1 > \omega_0$. The ground state and the ionized states are coupled by a time-dependent perturbative term such that the total Hamiltonian reads

$$\hat{H} = \hat{H}_0 + \hat{V}(t), \tag{4}$$

where

$$\hat{H}_0 = \hbar\omega_0|g\rangle\langle g| + \sum_I \hbar(\omega_1 + \omega_I)|I\rangle\langle I|,$$
$$\hat{V}(t) = V_0 \sum_I (|I\rangle\langle g|e^{-i\omega t} + \text{h.c.}).$$

For an atom initially prepared in its ground state $|g\rangle$ at time $t = 0$, determine the rate Γ at which the atom is ionized by using the expression

$$w_{i\to f}(T) = \frac{1}{i\hbar}\int_{-T/2}^{+T/2} \mathrm{d}\tau \langle f|\hat{V}(\tau)|i\rangle e^{i(E_f - E_i)\tau/\hbar}, \tag{5}$$

where $w_{i\to f}$ is the transition amplitude over a time T from state $|i\rangle$ at energy E_i to a state $|f\rangle$ at energy E_f due to a perturbation $\hat{V}(t)$.
Hint: You shall go to the continuous limit where $\sum_I \to \int \mathrm{d}E_I \rho(E_I)$, with $\rho(E)$ being the density of state at energy E.

(2 points)

Exercise 18 *Time-dependent perturbation theory*

Let us consider a two-level system described over its energy eigenbasis by the states $\{|g\rangle, |e\rangle\}$, such that its Hamiltonian reads

$$\hat{H}_0 = E_e|e\rangle\langle e| + E_g|g\rangle\langle g|, \tag{6}$$

where $E_e > E_g$. Adding a time-dependent perturbation such that $\hat{H} = \hat{H}_0 + \lambda\hat{V}(t)$, the perturbating term $\hat{V}(t)$ reads

$$\hat{V}(t) = \Omega_0 \left(|g\rangle\langle e|e^{i\omega t} + |e\rangle\langle g|e^{-i\omega t}\right). \tag{7}$$

At initial time $t = 0$, we assume that the system lies in state $|\psi_0\rangle = |g\rangle$.

a) Determine in first-order perturbation theory the probability amplitude of a transition to state $|e\rangle$ at time t. Show that it takes the form

$$w(t) = \langle e|\tilde{\psi}\rangle_t = -\frac{\lambda\Omega_0}{E_e - E_g - \hbar\omega}\left(e^{\frac{i}{\hbar}t(E_e - E_g - \hbar\omega)} - 1\right), \tag{8}$$

where $|\tilde{\psi}\rangle_t = e^{\frac{i}{\hbar}t\hat{H}_0}|\psi\rangle_t$ is the state of the system in the interaction picture.

(2 points)

b) In the following, we define the transition detuning as $\delta = E_e - E_g - \hbar\omega$. The dynamics that we have just treted perturbatively can actually be solved exactly by the means of a change of reference frame $|\psi\rangle = \hat{U}(t)|\psi'\rangle$, where

$$\hat{U}(t) = e^{-i\omega t/2}|e\rangle\langle e| + e^{+i\omega t/2}|g\rangle\langle g|. \tag{9}$$

Using this change of reference frame, show that the time-evolution of the two-level system is described by a state of the form $|\psi'\rangle_t = c_e(t)|e\rangle + c_g(t)|g\rangle$, where

$$c_g(t) = \cos\left(\frac{\sqrt{\delta^2 + 4\lambda^2\Omega_0^2}}{2\hbar}t\right) + \frac{i\delta}{\sqrt{\delta^2 + 4\lambda^2\Omega_0^2}}\sin\left(\frac{\sqrt{\delta^2 + 4\lambda^2\Omega_0^2}}{2\hbar}t\right) \tag{10}$$

$$c_e(t) = -\frac{2i\lambda\Omega_0}{\sqrt{\delta^2 + 4\lambda^2\Omega_0^2}}\sin\left(\frac{\sqrt{\delta^2 + 4\lambda^2\Omega_0^2}}{2\hbar}t\right). \tag{11}$$

Perform a Taylor expansion in λ of the transition amplitude $c_e(t)$ and compare it to the result obtained with the perturbation theory.

(4 points)