Theoretical physics V
Sheet 5

SoSe 2024 **Due for the 23.05.2024**

Exercise 13 *Goeppert-Mayer transformation*

Let us consider an electron of mass m and charge $q = -e$ set in a position-dependent potential V_0. The interaction of the electron with an external electric field $\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$ is described by the minimal coupling Hamiltonian

$$\hat{H} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A}\right)^2 + V_0(\hat{\boldsymbol{r}}). \tag{1}$$

In the following, we will assume that we are in the Coulomb gauge ($\boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$) such that the momentum operator $\hat{\boldsymbol{p}}$ commutes with the vector potential $\boldsymbol{A}$.

a) Considering a time-dependent unitary transformation $\hat{U}(t)$, such that a state $|\tilde{\psi}\rangle$ transforms into $|\psi\rangle$ following the relation $|\psi\rangle = \hat{U}|\tilde{\psi}\rangle$. Considering that the evolution of $|\psi\rangle$ is governed by the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \hat{H}|\psi\rangle, \tag{2}$$

determine the effective Hamiltonian $\hat{H}'$ describing the evolution of the transformed state $|\tilde{\psi}\rangle$.

(1 point)

b) We introduce the unitary operator $\hat{U}(t) = \exp\left(\frac{i}{\hbar c}\hat{\boldsymbol{d}} \cdot \boldsymbol{A}\right)$, where $\hat{\boldsymbol{d}} = -e\hat{\boldsymbol{r}}$ is the dipolar moment of the electron. Determine the expression of the commutator $\left[\hat{\boldsymbol{p}}, \hat{U}\right]$.

Hint: You may use that $\left[\hat{p}_i, (\hat{r}_j)^k\right] = -i\hbar k(\hat{r}_j)^{k-1}\delta_{ij}$, with $k \in \mathbb{N}$.

(2 points)

c) Show then that the effective Hamiltonian associated to the Goeppert-Meyer transformation $\hat{U}(t)$ introduced in the previous question reads

$$\hat{H}' = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + V_0(\hat{\boldsymbol{r}}) - \hat{\boldsymbol{d}} \cdot \boldsymbol{E}, \tag{3}$$

which is pivotal to the description of the light-matter interaction.

(2 points)

Exercise 14 *Charge conjugation and antiparticles*

We remind that the solutions for the Dirac equation for a free particle of mass m moving with momentum $\boldsymbol{p}$ take the form $\psi^{(j)}$ such that:

$$\psi^{(j)}(\boldsymbol{x},t) = \mathcal{N} u^{(j)}(\boldsymbol{p}) e^{i(\boldsymbol{p}\cdot\boldsymbol{x}\ -Et)/\hbar} \text{ with } \mathcal{N} = \sqrt{(|E|+mc^2)/(2|E|V)}, \tag{4}$$

where the spinors $u^{(j)}$ take the form

$$u^{(1)}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)}(\boldsymbol{p}) = \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{E+mc^2} \\ -\dfrac{cp_z}{E+mc^2} \end{pmatrix}$$

$$u^{(3)}(\boldsymbol{p}) = \begin{pmatrix} -\dfrac{cp_z}{|E|+mc^2} \\ -\dfrac{c(p_x+ip_y)}{|E|+mc^2} \\ 1 \\ 0 \end{pmatrix}, \quad u^{(4)}(\boldsymbol{p}) = \begin{pmatrix} -\dfrac{c(p_x-ip_y)}{|E|+mc^2} \\ \dfrac{cp_z}{|E|+mc^2} \\ 0 \\ 1 \end{pmatrix},$$

with E standing for the energy of the particle and V for the volume available to the particle. Let us consider the charge conjugate of the negative-energy solutions of the Dirac equations, namely $\phi(\boldsymbol{x},t) = \mathcal{C}(\psi^{(3,4)}(\boldsymbol{x},t))^*$, where $\mathcal{C} = -i\gamma^2$. Explicitly compute the form of ϕ for both cases $j = 3,4$ and compare them to the positive-energy solutions of the Dirac equation.

(2 points)

Exercise 15 *Klein paradox*

In the following, we consider the case of a plane wave electron of energy E and momentum $\boldsymbol{p} = p\boldsymbol{e}_z$ incident on a potential barrier $V(\boldsymbol{x}) = V\theta(z)$, $\theta(z)$ is the Heaviside distribution. The wavefunction of the incident electron ψ_i therefore takes the form

$$\psi_i(z,t) = \begin{pmatrix} 1 \\ 0 \\ \dfrac{pc}{E+mc^2} \\ 0 \end{pmatrix} e^{i(pz-Et)/\hbar}. \tag{5}$$

Based on the intuition built on non-relativistic quantum mechanics, we expect the plane wave to be partially transmitted and reflected by the potential barrier.

a) Determine the energy E' and the momentum p' of the electron for $z > 0$ as a function of E, m and V.

(1 point)

b) Based on continuity conditions, the reflected and transmitted waves are linked via the relation at $z = 0$,

$$\psi_i(\boldsymbol{x}, t) + r\psi_r(\boldsymbol{x}, t) = t\psi_t(\boldsymbol{x}, t),$$

where r and t are the reflection and transmition coefficients. Determine the expression of r and t as a function of the adimensional quantity ζ

$$\zeta = \frac{p'}{p} \frac{(E + mc^2)}{(E' + mc^2)}. \tag{6}$$

Note that in general r, t and ζ are complex numbers.

(3 points)

c) Compute the z-component of the vector current $j^3 = \psi^\dagger \alpha^3 \psi$ for the incident $\boldsymbol{j}_i$, reflected $\boldsymbol{j}_r$ and transmitted $\boldsymbol{j}_t$ currents. Verify the conservation of the currents: $j_i^3 + j_r^3 = j_t^3$.

(2 points)

d) Discuss the behavior of the ratio j_r^3/j_i^3 in three different cases:

1- For $E' > mc^2$;

2- For $mc^2 > E' > -mc^2$;

3- And for $E' < -mc^2$.

(3 points)

This third case leads to a seemingly paradoxical result discovered by Klein in 1929. It is however explained by the stimulated production of electron-positron pairs at the edge of the potential step by the incident lane wave: while positrons propagate to the right across the barrier, additional electrons are reflected to the left, leading to the anomalous result of Klein.