TPV

SoSe 2018 **Blatt 12** **12-13/07/2018**

Exercise 23 *Spontaneous emission*

We want to study the spontaneous of an hydrogen atom where we also want to take into account the photon recoil.

a) Give the non-relativistic Hamiltonian operator in the dipole approximation.

(*2 Points*)

b) The initial state is given by $|i\rangle = |\mathbf{P}, e; vac\rangle$, where $\mathbf{P}$ is the center of mass's momentum, $|e\rangle$ is the electronic excited state and $|vac\rangle$ is the vacuum state for the electromagnetic field. The energy is given by $E_i = \mathbf{P}^2/2M + \hbar\omega_e$, with $M = m_1 + m_2$ the total mass and $\omega_0 = \omega_e - \omega_g$ is the transition frequency. Calculate the matrix element $V_{fi} = \langle f|\hat{H}^{(1)}_{int}|i\rangle$, $|f\rangle = |\mathbf{P}', g; 1_\lambda\rangle$ with $E_f = \mathbf{P}'^2/2M + \hbar\omega_\lambda$. Here $|g\rangle$ is the vacuum state of the atom after the emission of the photon and $|1_\lambda\rangle$ is a one-photon state in mode λ.

(*2 Points*)

c) Calculate the transition amplitude $S_{fi} = -2\pi i V_{fi}\delta^{(T)}(E_f - E_i)$ to first order with

$$\delta^{(T)}(E_f - E_i) = \frac{1}{2\pi\hbar}\int_{-T/2}^{+T/2} \mathrm{d}\tau\, e^{i(E_f - E_i)\tau/\hbar} = \frac{1}{\pi}\frac{\sin\left[(E_f - E_i)T/2\hbar\right]}{E_f - E_i}. \tag{1}$$

How does the frequency of the emitted photon depends on the direction of the emission with respect to the motion? When can the effect of the motion be neglected?

(*2 Points*)

From now on neglect the center of mass motion.

d) Show that

$$\sum_f \frac{|S_{fi}|^2}{T} = \frac{2\pi}{\hbar}\int_0^{+\infty} \mathrm{d}E_f \rho(E_f)\left|\overline{\langle f|\hat{H}^{(1)}_{int}|i\rangle}\right|^2 \delta(E_f - E_i) \tag{2}$$

in the infinite interaction time and quantisation volume limit, where

$$\left|\overline{\langle f|\hat{H}^{(1)}_{int}|i\rangle}\right|^2 = \frac{2\pi\hbar\omega_\lambda}{V}|\mathbf{d}_{ge}|^2 \tag{3}$$

and

$$\rho(E) = \frac{VE^2}{(2\pi c\hbar)^3}\int \mathrm{d}\Omega \sum_{\boldsymbol{\epsilon}_\lambda \perp \mathbf{n}} \frac{|\mathbf{d}_{ge}\cdot\boldsymbol{\epsilon}_\lambda|^2}{|\mathbf{d}_{ge}|^2} \tag{4}$$

where $\mathbf{d}_{ge} = \langle g|\mathbf{d}|e\rangle$.

(*4 Points*)

e) Show that $\rho(E)$ can be written as

$$\rho(E) = \frac{VE^2}{(2\pi c\hbar)^3}\frac{8\pi}{3}. \tag{5}$$

Hint: In order to obtain the final expression, use that

$$\sum_{\boldsymbol{\epsilon}_\lambda \perp \mathbf{k}_\lambda} |\boldsymbol{\epsilon}_\lambda \cdot \mathbf{d}_{ge}|^2 = |\mathbf{d}_{ge}|^2(1-\cos^2\theta) \tag{6}$$

where, for fixed $\mathbf{k}_\lambda$*,* θ *is the angle between* $\mathbf{d}_{ge}$ *and the* $\mathbf{k}_\lambda$ *vector.*

(3 Points)

f) Derive that

$$\Gamma = \frac{4\omega_0^3|\mathbf{d}_{ge}|^2}{3\hbar c^3}. \tag{7}$$

(1 Point)

Exercise 24 *Lagrangian of the free Dirac fields*

The basic-free field Lagrangian density from which the field equation for the Dirac fields can be derived is:

$$\hat{\mathcal{L}} = -c\hbar\hat{\bar{\psi}}_\alpha(\gamma_\mu)_{\alpha\beta}\partial_\mu\hat{\psi}_\beta - mc^2\delta_{\alpha\beta}\hat{\bar{\psi}}_\alpha\hat{\psi}_\beta \tag{8}$$

where $\hat{\bar{\psi}} = \hat{\psi}^\dagger(\gamma_4)$ and each of the four components of $\hat{\psi}$ and $\hat{\bar{\psi}}$ is to be regarded as an independent field variable.

a) Show that the four Euler-Lagrange equations obtained by varying $\hat{\bar{\psi}}_\alpha$ can be summarized as the single Dirac equation (in the van der Waerden's form)

$$\Big((\gamma_\mu)_{\alpha\beta}\,\partial_\mu + \frac{mc}{\hbar}\delta_{\alpha\beta}\Big)\hat{\psi}_\beta = 0. \tag{9}$$

(2 Points)

b) To obtain the field equation for $\overline{\hat{\psi}}$, make the replacement

$$-c\hbar\hat{\bar{\psi}}_\alpha(\gamma_\mu)_{\alpha\beta}\partial_\mu\hat{\psi}_\beta \to c\hbar\Big(\partial_\mu\hat{\bar{\psi}}_\alpha\Big)(\gamma_\mu)_{\alpha\beta}\hat{\psi}_\beta \tag{10}$$

and argue why it is justified. Obtain

$$-\partial_\mu\hat{\bar{\psi}}_\beta(\gamma_\mu)_{\beta\alpha} + \frac{mc}{\hbar}\hat{\bar{\psi}}_\alpha = 0 \tag{11}$$

by varying $\hat{\psi}_\alpha$.

(1 Point)

c) Determine the canonical conjugate momentum $\hat{\Pi}_\beta$

$$\hat{\Pi}_\beta = \frac{\partial \hat{\mathcal{L}}}{\partial(\partial \hat{\psi}_\beta / \partial t)}. \tag{12}$$

What is the canonical momentum conjugate to $\hat{\overline{\psi}}_\beta$?

(1 Point)

d) Determine that the Hamiltonian density reads

$$\hat{\mathcal{H}} = \hat{\psi}^\dagger \Big(-i\hbar c\, \hat{\boldsymbol{\alpha}} \cdot \boldsymbol{\nabla} + \hat{\alpha}_t mc^2 \Big) \hat{\psi}. \tag{13}$$

(1 Point)