Supplementary material for "Entropy-regularized mean-variance portfolio optimization with jumps"

Christian Bender[*] Nguyen Tran Thuan[†‡]

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Abstract

This document contains supplementary materials for the main article [1]. All notations used here are in accordance with that in [1].

1 Integrability for solutions of SDEs with jumps

Although the following fact can be easily extended to a multidimensional setting, however, we formulate it in the one-dimensional case for the sake of simplicity.

Lemma 1.1. *Let $\xi = (\xi_t)_{t\in[0,T]}$ be càdlàg and adapted with $\|\xi\|^2_{\mathcal{S}_2([0,T])} := \mathbb{E}[\sup_{0\le t\le T} \xi_t^2] < \infty$. Assume that $\mathrm{d}Z_t = \phi_t \mathrm{d}t + \mathrm{d}K_t$, where $K = (K_t)_{t\in[0,T]}$ is a càdlàg $\mathbf{L}_2(\mathbb{P})$-martingale satisfying $\mathrm{d}\langle K, K\rangle_t = \eta_t^2 \mathrm{d}t$, where η and ϕ are progressively measurable with $\sup_{0<t<T} \eta_t^2 + \int_0^T \phi_t^2 \mathrm{d}t \le C$ a.s. for some (non-random) constant $C > 0$. Then, for a Lipschitz function $\sigma\colon \mathbb{R} \to \mathbb{R}$, the SDE*

$$X_t = \xi_t + \int_0^t \sigma(X_{u-})\mathrm{d}Z_u, \quad X_0 = \xi_0 = x_0 \in \mathbb{R}, \tag{1.1}$$

has a unique càdlàg strong solution $X = (X_t)_{t\in[0,T]}$ satisfying $\mathbb{E}[\sup_{0\le t\le T} X_t^2] \le C' < \infty$ for some constant $C' = C'(\|\xi\|_{\mathcal{S}_2([0,T])}, T, \sigma, C) > 0$.

Proof. Due to [4, Theorem V.3.7], the SDE (1.1) has a unique càdlàg and adapted solution X. For $n \ge 1$ we define $\tau_n := \inf\{t > 0 : |X_t| \ge n\} \wedge T$. Then τ_n is a stopping time with $|X_{(t\wedge\tau_n)-}| \le n$ for $t \in [0, T]$. It is known that, see, e.g., [4, Theorem II.5.12], a.s.,

$$X_{t\wedge\tau_n} = \xi_{t\wedge\tau_n} + \int_0^{t\wedge\tau_n} \sigma(X_{u-})\mathrm{d}Z_u = \xi_{t\wedge\tau_n} + \int_0^t \mathbb{1}_{(0,\tau_n]}(u)\sigma(X_{u-})\mathrm{d}K_u + \int_0^t \mathbb{1}_{(0,\tau_n]}(u)\sigma(X_{u-})\phi_u \mathrm{d}u$$

so that the triangle inequality, Itô's isometry, and Hölder's inequality yield

$$\begin{aligned}
\frac{1}{3}\mathbb{E}[X^2_{t\wedge\tau_n}] &\le \mathbb{E}[\xi^2_{t\wedge\tau_n}] + \mathbb{E}\left[\int_0^t \mathbb{1}_{(0,\tau_n]}(u)\sigma(X_{u-})^2\eta_u^2 \mathrm{d}u\right] + \mathbb{E}\left[\left|\int_0^t \mathbb{1}_{(0,\tau_n]}(u)\sigma(X_{u-})\phi_u \mathrm{d}u\right|^2\right] \\
&\le \|\xi\|^2_{\mathcal{S}_2([0,T])} + C\mathbb{E}\left[\int_0^t \mathbb{1}_{(0,\tau_n]}(u)\sigma(X_{u-})^2 \mathrm{d}u\right] \\
&\le \|\xi\|^2_{\mathcal{S}_2([0,T])} + 2CT\sigma(0)^2 + 2C|\sigma|^2_{\mathrm{Lip}}\mathbb{E}\left[\int_0^t \mathbb{1}_{(0,\tau_n]}(u)X^2_{u-} \mathrm{d}u\right] \\
&= \alpha + \beta\mathbb{E}\left[\int_0^t \mathbb{1}_{(0,\tau_n]}(u)X^2_{u-} \mathrm{d}u\right]
\end{aligned}$$

for $\alpha := \|\xi\|^2_{\mathcal{S}_2([0,T])} + 2CT\sigma(0)^2$, $\beta := 2C|\sigma|^2_{\mathrm{Lip}}$, and $|\sigma|_{\mathrm{Lip}} := \sup_{x\neq y} \frac{|\sigma(x)-\sigma(y)|}{|x-y|}$. Since

$$\mathbb{1}_{(0,\tau_n]}(u)X^2_{u-} \le n^2, \quad u \in [0, T],$$

[*]Department of Mathematics, Saarland University, Germany. Email: bender@math.uni-saarland.de
[†]Department of Mathematics, Saarland University, Germany. Email: nguyen@math.uni-saarland.de
[‡]Department of Mathematics, Vinh University, Vinh, Nghe An, Viet Nam. Email: thuan.tr.nguyen@gmail.com

it implies that

$$\mathbb{E}[X_{t\wedge\tau_n}^2] \le 3\alpha + 3\beta T n^2, \quad \forall t\in[0,T].$$

Moreover, as X has càdlàg paths, we get for all $t\in[0,T]$ that

$$\frac{1}{3}\mathbb{E}[X_{t\wedge\tau_n}^2] \le \alpha + \beta\mathbb{E}\left[\int_0^t \mathbb{1}_{(0,\tau_n]}(u)X_u^2\mathrm{d}u\right] \le \alpha+\beta\mathbb{E}\left[\int_0^t X_{u\wedge\tau_n}^2\mathrm{d}u\right] = \alpha+\beta\int_0^t \mathbb{E}[X_{u\wedge\tau_n}^2]\mathrm{d}u.$$

Applying Gronwall's lemma yields $\mathbb{E}[X_{t\wedge\tau_n}^2]\le 3\alpha\mathrm{e}^{3\beta T}$ for all $t\in[0,T]$, $n\ge 1$. Since $(\tau_n)_{n\ge1}$ is eventually constant T a.s., sending $n\to\infty$ and using Fatou's lemma we obtain

$$\mathbb{E}[X_t^2]\le \liminf_{n\to\infty}\mathbb{E}[X_{t\wedge\tau_n}^2]\le 3\alpha\mathrm{e}^{3\beta T}, \quad \forall t\in[0,T].$$

As a consequence, $\int_0^\cdot \sigma(X_{t-})\mathrm{d}K_t$ is an $\mathbf{L}_2(\mathbb{P})$-martingale. Therefore, applying Doob's maximal inequality for the martingale part we get

$$\mathbb{E}\left[\sup_{0\le t\le T} X_t^2\right] \le \|\xi\|_{\mathcal{S}_2([0,T])}^2 + 4\mathbb{E}\left[\int_0^T \sigma(X_{t-})^2\eta_t^2\mathrm{d}t\right] + \mathbb{E}\left[\int_0^T\sigma(X_{t-})^2\mathrm{d}t\int_0^T\phi_t^2\mathrm{d}t\right]<\infty,$$

which completes the proof. □

2 Explicit expression for the optimal wealth and Lagrange multiplier

We give in this part a closed-form representation of the optimal wealth X^* and the respective Lagrange multiplier $\hat{w}$ when the condition "$\Delta Z\ne 1$ on $[0,T]$" in [1, Proposition 4.12] fails to hold. Let us impose the assumptions of [1, Theorem 4.10]. For Z given in that theorem, we write

$$-(X_{s-}^*-\hat{w})\mathrm{d}Z_s = (X_{s-}^*-\hat{w})\mathrm{d}(-Z_s)$$

and follow [4, Exercise V.27] to define the sequence of stopping times $\{\tau_n\}_{n\ge1}$ by setting

$$\tau_0:=0, \quad \tau_n:=\inf\{\tau_{n-1}<s\le T: 1-\Delta Z_s=0\}, \quad n\ge 1.$$

Note that τ_n is non-decreasing and tends to ∞ a.s. as $n\to\infty$. Then the solution X^* of [1, Eq. (4.23)] is

$$X_r^* = \hat{w} + \sum_{n=1}^{\infty} X_r^{*,n-1}\mathbb{1}_{[\tau_{n-1},\tau_n)\cap[0,T]}(r), \quad r\in[0,T], \tag{2.1}$$

where we conventionally set $[\infty,\infty):=\emptyset$. In (2.1), $X^{*,n-1}=(X_r^{*,n-1})_{r\in[0,T]}$ is given by

$$X_r^{*,n-1} := \left[(x_0-\hat{w})\mathbb{1}_{\{n=1\}} + \sqrt{\frac{\lambda}{2}}\left(\Delta M_{\tau_{n-1}} + \int_{\tau_{n-1}}^r \frac{\mathrm{d}M_s}{U_{s-}^{n-1}} + \int_{\tau_{n-1}}^r \frac{\mathrm{d}[M,Z]_s}{U_{s-}^{n-1}}\right)\right]U_r^{n-1}\mathbb{1}_{[\tau_{n-1},\tau_n)\cap[0,T]}(r).$$

The process $U^{n-1}=(U_r^{n-1})_{r\in[0,T]}$ is defined by

$$\begin{aligned} U_r^{n-1} &:= \begin{cases} 1 & \text{if } r\le\tau_{n-1} \\ \mathrm{e}^{-Z_r+Z_{\tau_{n-1}}-\frac{1}{2}\int_{\tau_{n-1}}^r \mathrm{d}[Z,Z]_s^c}\displaystyle\prod_{\tau_{n-1}<s\le r}(1-\Delta Z_s)\mathrm{e}^{\Delta Z_s} & \text{if } r>\tau_{n-1}\end{cases} \\ &= \mathcal{E}(-Z+Z^{\tau_{n-1}})_r, \end{aligned}$$

where $\mathcal{E}(-Z+Z^{\tau_{n-1}})=(\mathcal{E}(-Z+Z^{\tau_{n-1}})_r)_{r\in[0,T]}$ denotes the Doléans–Dade exponential of $-Z+Z^{\tau_{n-1}}$, see [4, Section II.8], and where $Z^{\tau_{n-1}}$ is the process Z stopped at τ_{n-1}, i.e. $Z_t^{\tau_{n-1}}:=Z_{t\wedge\tau_{n-1}}$.

We now calculate the Lagrange multiplier $\hat{w}$ using the constraint $\mathbb{E}[X_T^*]=\hat{z}$. One first has

$$\mathrm{d}\mathcal{E}(-Z+Z^{\tau_{n-1}})_s = \mathcal{E}(-Z+Z^{\tau_{n-1}})_{s-}\mathrm{d}(-Z_s+Z_s^{\tau_{n-1}}), \quad \mathcal{E}(-Z+Z^{\tau_{n-1}})_0=1,$$

and the *conditional* quadratic variation[1] of $-Z + Z^{\tau_{n-1}}$, which is computed by

$$\begin{aligned}
&\langle -Z + Z^{\tau_{n-1}}, -Z + Z^{\tau_{n-1}} \rangle \\
&= \left\langle \int_0^\cdot \mathbb{1}_{(\tau_{n-1},T]}(s)(\mathcal{M}_\alpha^\top \mathcal{S}_\alpha^{-1})(s, Y_{s-})\mathrm{d}Y_s, \int_0^\cdot \mathbb{1}_{(\tau_{n-1},T]}(s)(\mathcal{M}_\alpha^\top \mathcal{S}_\alpha^{-1})(s, Y_{s-})\mathrm{d}Y_s \right\rangle \\
&= \int_0^\cdot \mathbb{1}_{(\tau_{n-1},T]}(s)(\mathcal{M}_\alpha^\top \mathcal{S}_\alpha^{-1}\Sigma\mathcal{S}_\alpha^{-1}\mathcal{M}_\alpha)(s, Y_{s-})\mathrm{d}s,
\end{aligned}$$

has uniformly bounded integrand over $(n, s) \in \mathbb{N}\times(0, T)$ a.s. by [1, Eq. (4.20)]. Then applying Lemma 1.1 yields $\sup_{n\geq 1} \mathbb{E}[|U_T^{n-1}|^2] = \sup_{n\geq 1} \mathbb{E}[|\mathcal{E}(-Z + Z^{\tau_{n-1}})_T|^2] < \infty$. In particular, for $n = 1$ we can define

$$d_{(2.2)} := \mathbb{E}[\mathcal{E}(-Z)_T \mathbb{1}_{[0,\tau_1)\cap[0,T]}(T)] = \mathbb{E}[\mathcal{E}(-Z)_T \mathbb{1}_{\{\tau_1=\infty\}}] \in \mathbb{R}. \tag{2.2}$$

Moreover, using Lemma 1.1 again we assert that X^* is a square integrable process which together with (2.1) and (2.2) then imply that

$$d_{(2.3)} := \mathbb{E}\Bigg[\sum_{n=1}^\infty \left(\Delta M_{\tau_{n-1}} + \int_{\tau_{n-1}}^T \frac{\mathrm{d}M_s}{U_{s-}^{n-1}} + \int_{\tau_{n-1}}^T \frac{\mathrm{d}[M,Z]_s}{U_{s-}^{n-1}}\right) U_T^{n-1}\mathbb{1}_{[\tau_{n-1},\tau_n)\cap[0,T]}(T)\Bigg] \tag{2.3}$$

finitely exists. Now we let $r = T$ and take the expectation both sides of (2.1) to get

$$\hat{z} = \hat{w} + d_{(2.2)}(x_0 - \hat{w}) + \sqrt{\frac{\lambda}{2}} d_{(2.3)},$$

If $d_{(2.2)} \neq 1$, then the Lagrange multiplier $\hat{w}$ is calculated by

$$\hat{w} = \frac{\hat{z} - \sqrt{\frac{\lambda}{2}} d_{(2.3)} - d_{(2.2)} x_0}{1 - d_{(2.2)}}.$$

3 Some auxiliary results for Lévy processes

All Lévy processes below are considered with the canonical truncation function $h(x) = x\mathbb{1}_{\{\|x\|\leq 1\}}$.

Lemma 3.1. *L is a D-dimensional Lévy process if and only if $u^\top L$ is a 1-dimensional Lévy process for all $u \in \mathbb{R}^D$. Moreover, L has characteristic (b, A, ν) if and only if $u^\top L$ has characteristic $(b_u, u^\top Au, \nu \circ \{y \mapsto u^\top y\}^{-1})$ where $b_u := u^\top b - \int_{u^\top y\neq 0} u^\top y(\mathbb{1}_{\{\|y\|\leq 1\}} - \mathbb{1}_{\{|u^\top y|\leq 1\}})\nu(\mathrm{d}y)$ for all $u \in \mathbb{R}^D$.*

Proof. It is obvious that L has càdlàg paths a.s. if and only if $u^\top L$ has càdlàg paths a.s. for all $u \in \mathbb{R}^D$. We now verify the equivalence regarding distributional properties. Let $\mathcal{F}_t^L := \sigma\{L_s : s \leq t\}$. Assume that L is a D-dimensional Lévy process with characteristic (b, A, ν). Then it follows from [3, Theorem 3.1] that, for any $s \leq t$ and $x \in \mathbb{R}$, a.s.,

$$\mathbb{E}\big[\mathrm{e}^{\mathrm{i}xu^\top(L_t - L_s)}\big|\mathcal{F}_s^L\big] = \mathrm{e}^{-(t-s)\kappa(xu)}.$$

By a change of variables we have

$$\begin{aligned}
\kappa_u(x) := \kappa(xu) &= -\mathrm{i}xu^\top b + \frac{x^2 u^\top Au}{2} - \int_{y\neq 0} (\mathrm{e}^{\mathrm{i}xu^\top y} - 1 - \mathrm{i}xu^\top y\mathbb{1}_{\{\|y\|\leq 1\}})\nu(\mathrm{d}y) \\
&= -\mathrm{i}xu^\top b + \frac{x^2 u^\top Au}{2} - \int_{u^\top y\neq 0} (\mathrm{e}^{\mathrm{i}xu^\top y} - 1 - \mathrm{i}xu^\top y\mathbb{1}_{\{\|y\|\leq 1\}})\nu(\mathrm{d}y) \\
&= -\mathrm{i}x\left(u^\top b - \int_{u^\top y\neq 0} u^\top y(\mathbb{1}_{\{\|y\|\leq 1\}} - \mathbb{1}_{\{|u^\top y|\leq 1\}})\nu(\mathrm{d}y)\right) \\
&\quad + \frac{x^2 u^\top Au}{2} - \int_{z\neq 0} (\mathrm{e}^{\mathrm{i}xz} - 1 - \mathrm{i}xz\mathbb{1}_{\{|z|\leq 1\}})\nu \circ \{y \mapsto u^\top y\}^{-1}(\mathrm{d}z).
\end{aligned}$$

Hence, applying [3, Theorem 3.1] once more shows that $u^\top L$ is a Lévy process with the characteristic exponent κ_u. The converse implication is straightforward by choosing $x = 1$. ☐

[1] See, e.g., [4, Chapter III, p.124].

Lemma 3.2. *Let $D, D' \in \mathbb{N}$. Assume that W is a D-dimensional Gaussian Lévy process and L is a D'-dimensional purely non-Gaussian Lévy process, both defined on the same probability space. Then W and L are independent.*

Proof. ***Step 1.*** We prove that, for any $u \in \mathbb{R}^D$, $v \in \mathbb{R}^{D'}$, two processes $u^\top W$ and $v^\top L$ are independent. Indeed, it is obvious that $u^\top W$ is a Gaussian Lévy process, and $v^\top L$ is a purely non-Gaussian Lévy process due to Lemma 3.1. Denote by $[X, Y]$ the quadratic covariation of two càdlàg real semimartingales X, Y (see, e.g., [4, p.66] or [2, Definition 8.2]). By the bilinearity of quadratic covariation, we get

$$[u^\top W, v^\top L] = \left[\sum_{d=1}^{D} u^{(d)} W^{(d)}, \sum_{d'=1}^{D'} v^{(d')} L^{(d')}\right] = \sum_{d=1}^{D}\sum_{d'=1}^{D'} u^{(d)} v^{(d')} [W^{(d)}, L^{(d')}].$$

Since $W^{(d)}$ is continuous and $L^{(d')}$ is purely non-Gaussian, both are Lévy processes null at 0, it implies that $[W^{(d)}, L^{(d')}] = 0$. Hence, $[u^\top W, v^\top L] = 0$. We then apply [2, Theorem 11.43] to get the independence of $u^\top W$ and $v^\top L$ as desired.

Step 2. By choosing a common refinement of partitions, it suffices to prove that $(W_{t_1}, \ldots, W_{t_n} - W_{t_{n-1}})$ is independent of $(L_{t_1}, \ldots, L_{t_n} - L_{t_{n-1}})$ for all $0 = t_0 < t_1 < \cdots < t_n$, $n \in \mathbb{N}$. Let $\{u_k\}_{k=1}^n \subset \mathbb{R}^D$ and $\{v_k\}_{k=1}^n \subset \mathbb{R}^{D'}$ arbitrarily. One has

$$\begin{aligned} I_{(3.1)} &:= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^n u_k^\top (W_{t_k} - W_{t_{k-1}}) + \mathrm{i}\sum_{k=1}^n v_k^\top (L_{t_k} - L_{t_{k-1}})}\right] \\ &= \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^n \sum_{d=1}^D u_k^{(d)} (W_{t_k}^{(d)} - W_{t_{k-1}}^{(d)}) + \mathrm{i}\sum_{k=1}^n \sum_{d'=1}^{D'} v_k^{(d')} (L_{t_k}^{(d')} - L_{t_{k-1}}^{(d')})}\right]. \end{aligned} \tag{3.1}$$

For $d = 1, \ldots, D$, $d' = 1, \ldots, D'$ and $k = 1, \ldots, n$ we define, for $t \in [0, 1]$,

$$\check{W}_t^{(d,k)} := W_{(t_k - t_{k-1})t + t_{k-1}}^{(d)} - W_{t_{k-1}}^{(d)} \quad \text{and} \quad \check{L}_t^{(d',k)} := L_{(t_k - t_{k-1})t + t_{k-1}}^{(d')} - L_{t_{k-1}}^{(d')},$$

and set $\check{W}_t := (\check{W}_t^{(d,k)})_{1 \le d \le D, 1 \le k \le n} \in \mathbb{R}^{D \times n}$, $\check{L}_t := (\check{L}_t^{(d',k)})_{1 \le d' \le D', 1 \le k \le n} \in \mathbb{R}^{D' \times n}$. We now show that $\check{L} = (\check{L}_t)_{t \in [0,1]}$ is an $\mathbb{R}^{D' \times n}$-valued purely non-Gaussian Lévy process. For any $v \in \mathbb{R}^{D' \times n}$ and $t \in [0, 1]$, one has

$$\mathbf{tr}[v^\top \check{L}_t] = \sum_{k=1}^{n}\sum_{d'=1}^{D'} v^{(d',k)} \check{L}_t^{(d',k)} = \sum_{k=1}^{n}\sum_{d'=1}^{D'} v^{(d',k)} (L_{(t_k - t_{k-1})t + t_{k-1}}^{(d')} - L_{t_{k-1}}^{(d')}).$$

For each $k = 1, \ldots, n$, since $[0, 1] \ni t \mapsto L_{(t_k - t_{k-1})t + t_{k-1}} - L_{t_{k-1}}$ is a D'-dimensional purely non-Gaussian Lévy process, it follows from Lemma 3.1 that

$$[0, 1] \ni t \mapsto \sum_{d'=1}^{D'} v^{(d',k)} (L_{(t_k - t_{k-1})t + t_{k-1}}^{(d')} - L_{t_{k-1}}^{(d')})$$

is also a purely non-Gaussian Lévy process. Since L has independent increments, we infer that $(\mathbf{tr}[v^\top \check{L}_t])_{t \in [0,1]}$ is again a purely non-Gaussian Lévy process. Analogously, $(\check{W}_t)_{t \in [0,1]}$ is an $\mathbb{R}^{D \times n}$-valued Gaussian Lévy process without drift. By vectorization and applying ***Step 1*** we get that $(\mathbf{tr}[u^\top \check{W}_t])_{t \in [0,1]}$ is independent of $(\mathbf{tr}[v^\top \check{L}_t])_{t \in [0,1]}$ for any $u \in \mathbb{R}^{D \times n}$ and $v \in \mathbb{R}^{D' \times n}$. Therefore, choosing particularly $t = 1$ yields

$$I_{(3.1)} = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^n \sum_{d=1}^D u_k^{(d)} (W_{t_k}^{(d)} - W_{t_{k-1}}^{(d)})}\right] \mathbb{E}\left[\mathrm{e}^{\mathrm{i}\sum_{k=1}^n \sum_{d'=1}^{D'} v_k^{(d')} (L_{t_k}^{(d')} - L_{t_{k-1}}^{(d')})}\right],$$

which implies the desired conclusion. □

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