Sample problems for applicants to the study programme "Master in Mathematics" at Saarland University

1 Basic courses

Solid knowledge of material in basic courses is expected.

1.1 Linear Algebra

Recommended literature:

- M. Artin, *Algebra*, Prentice Hall, Englewood Cliffs, NJ, 1991
- S. Lang, *Linear algebra*, reprint of the third edition, Undergraduate Texts in Mathematics, Springer, New York, 1989

Problem 1

Let M be a set and $\emptyset \neq N \subsetneq M$. On the power set $\mathcal{P}(M)$ we define the following relations

$$\begin{aligned} U \sim V &:\Longleftrightarrow U \cap N = V \cap N \\ U\, R\, V &:\Longleftrightarrow U \setminus N \subseteq V. \end{aligned}$$

1. Show that $\sim$ is an equivalence relation on $\mathcal{P}(M)$.
2. Is R always reflexive, always symmetric or always transitive? Is it an equivalence relation, a partial order or neither?
3. In the case of $M := \{1, 2, 3, 4, 5\}$ and $N := \{1, 3, 5\}$ find all elements of the equivalence class of $\{1, 2, 3\} \in \mathcal{P}(M)$ with respect to $\sim$.
4. Which of the following rules define a map? If yes, is the map injective, surjective, bijective?

$$\begin{aligned} f_1 &: \mathcal{P}(M)/_\sim \to \mathcal{P}(M), \quad [U]_\sim \mapsto U \cup N \\ f_2 &: \mathcal{P}(M)/_\sim \to \mathcal{P}(N), \quad [U]_\sim \mapsto U \cap N. \end{aligned}$$

Problem 2

Given the matrix

$$A := \begin{pmatrix} 1 & 0 & 3 & 1 & 3 \\ 0 & 2 & -4 & 1 & 2 \\ 1 & 1 & 1 & -1 & -1 \\ 2 & 0 & 6 & -1 & 0 \end{pmatrix} \in \mathbb{R}^{4\times 5}$$

and the two vectors

$$b_1 := \begin{pmatrix} 2 \\ -3 \\ 3 \\ 7 \end{pmatrix} \text{ and } b_2 := \begin{pmatrix} 0 \\ 1 \\ -2 \\ -1 \end{pmatrix} \text{ in } \mathbb{R}^4.$$

1. Compute the set of solutions of the linear equations $Ax = b_1$ and $Ax = b_2$.
2. Let $U \leq \mathbb{R}^4$ be a subspace. Show that $V := \{v \in \mathbb{R}^5 \mid Av \in U\}$ is a subspace of $\mathbb{R}^5$. Let now U be the span of $b_1, b_2, b_1 - b_2$. Compute a basis of V.

Problem 3

We consider the permutations $\sigma := (1\,6)(3\,5)(2\,4)$ and $\tau := (1\,2\,5)(3\,6\,4) \in S_6$.

1. Show that σ and τ commute, i.e., $\sigma \circ \tau = \tau \circ \sigma$.
2. Let $U := \langle \sigma, \tau \rangle \leq S_6$ be the subgroup generated by σ and τ. Show that U is abelian, cyclic and generated by $\sigma \circ \tau$.

Problem 4

Let $(G, \star)$ and $(H, \circ)$ be two groups and $\varphi : G \to H$ a group homomorphism. The set

$$G \times H := \{(g, h) \mid g \in G, h \in H\}$$

is a group with the following group law $(g_1, h_1) \bullet (g_2, h_2) := (g_1 \star g_2, h_1 \circ h_2)$.

1. What is the neutral element of $G \times H$ and what is the inverse of $(g, h) \in G \times H$?
2. Show that $F := \{(g, \varphi(g)) \mid g \in G\}$ is a subgroup of $G \times H$.
3. In addition, let φ be surjective. Show that the map,

$$\psi : G \times H \to H, \quad (g, h) \mapsto \varphi(g) \circ h$$

is a group homomorphism if and only if H is abelian.

Problem 5

For any $n \in \mathbb{N}$ define the real matrix

$$A_n := \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & 2 & 2 & 0 & \dots & 0 \\ 0 & 1 & 3 & 3 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & n-1 & n-1 \\ 0 & \dots & \dots & 0 & 1 & n \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

1. Compute $\det(A_2)$ and $\det(A_3)$.
2. Find a formula for $\det(A_n)$ depending on n and prove it.
   Hint: Laplace expansion of the last row or column.

Problem 6

Given the matrix

$$A := \frac{1}{2}\begin{pmatrix} 1+\frac{1}{\sqrt{2}} & 1-\frac{1}{\sqrt{2}} & -1 \\ 1-\frac{1}{\sqrt{2}} & 1+\frac{1}{\sqrt{2}} & 1 \\ 1 & -1 & \sqrt{2} \end{pmatrix} \in \mathbb{R}^{3\times 3} \text{ and the vector } v := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{R}^3.$$

1. Show that A is an orthogonal matrix.
2. Show that v is an eigenvector of A. What is the corresponding eigenvalue?
3. The map $\phi_A : \mathbb{R}^3 \to \mathbb{R}^3, v \mapsto Av$ is a rotation in $\mathbb{R}^3$. Calculate the corresponding plane of rotation and the angle of rotation.
4. Consider the linear hull

$$V := \mathrm{Lin}(\begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ 1 \\ 3 \end{pmatrix}) \quad \text{and} \quad p := \begin{pmatrix} 1 \\ -3 \\ 1 \\ -5 \end{pmatrix}$$

   in $\mathbb{R}^4$. Calculate the distance of V and p with respect to the standard metric.

Problem 7

Calculate the Jordan normal form of

$$A := \begin{pmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \in \mathbb{C}^{4\times 4}.$$

Problem 8

1. Consider the vector space $\mathbb{R}^2 \otimes \mathbb{R}^3$ with basis

$$B := \{d_i \otimes e_j \mid i = 1, 2 \text{ and } j = 1, 2, 3\}.$$

   where d_1, d_2 and e_1, e_2, e_3 denote the standard basis vectors of $\mathbb{R}^2$, respeively $\mathbb{R}^3$ and $\otimes$ the tensor product. Express the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix} \otimes \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ as linear combination of the basis B.
2. Let V, W be two finite dimensional $\mathbb{R}$-vector spaces and V^*, W^* their duals. For $\alpha \in V^*, \beta \in W^*$ we define the map

$$\psi_{\alpha,\beta} : V \otimes W \to \mathbb{R}, \quad v \otimes w \mapsto \alpha(v)\beta(w).$$

   Show that $\psi_{\alpha,\beta}$ is a well defined linear map.
3. Show that there is no alternating, multilinear map $\mu : \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}^5$ whose image contains the standard basis $\{e_1, e_2, \ldots, e_5\} \subseteq \mathbb{R}^5$.

1.2 Real analysis

Recommended literature:

- K. R. Davidson and A. P. Donsig, *Real analysis and applications*, Undergraduate Texts in Mathematics, Springer, New York, 2010
- W. Rudin, *Principles of mathematical analysis*, third edition, International Series in Pure and Applied Mathematics, McGraw-Hill, New York-Auckland-Düsseldorf, 1976
- M. Spivak, *Calculus*, fourth edition, Publish or Perish, 2008

Problem 9

Determine which of the following series converge:

1. $\sum_{n=1}^{\infty} \frac{(n+1)!}{(2n)!}$.
2. $\sum_{n=1}^{\infty} \frac{n^2+\sin(n)}{2n}$.
3. $\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)}$.

Problem 10

Let $n \in \mathbb{N}$ be even and let $a, b > 0$. Let

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^n - ax - b.$$

Find all local extrema of f and show that f has exactly two zeros.

Problem 11

Let (a_n) be a sequence of real numbers. Assume that the subsequences (a_{2n}), (a_{2n+1}) and (a_{3n}) are all convergent. Is (a_n) itself convergent? Either provide a proof or a counterexample.

Problem 12

Let (a_n) be a decreasing sequence of non-negative real numbers. Show that $\sum_{n=0}^{\infty} a_n$ converges if and only if $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges.

Problem 13

Let X and Y be metric spaces and let $f : X \to Y$ be a uniformly continuous map.

(a) Let (x_n) be a Cauchy sequence in X. Show that $(f(x_n))$ is a Cauchy sequence in Y.

(b) Show that the conclusion of (a) may fail if f is merely assumed to be continuous.

Problem 14

Let X be a metric space. We say that a family $(A_i)_{i \in I}$ of subsets of X has the finite intersection property if for every finite subset $J \subset I$, we have $\bigcap_{j \in J} A_j \neq \emptyset$.

Show that the following assertions are equivalent:

(i) X is compact;

(ii) Whenever $(A_i)_{i\in I}$ is a family of closed subsets of X satisfying the finite intersection property, then $\bigcap_{i\in I} A_i \neq \emptyset$.

Problem 15

Let $I \subset \mathbb{R}$ be an open interval and let

$$f : I \times I \to \mathbb{R}, \quad (x, y) \mapsto f(x, y),$$

be continuous and continuously partially differentiable with respect to y. Let $a \in I$ and define

$$F : I \to \mathbb{R}, \quad F(y) = \int_a^y f(x, y)\, dx.$$

Show that F is differentiable with

$$F'(y) = f(y, y) + \int_a^y \frac{\partial f}{\partial y}(x, y)\, dx \quad \text{for all } y \in I.$$

1.3 Numerics

Recommended literature:

- J. Stoer and R. Z. Bulirsch, *Introduction to numerical analysis*, translated from the German by R. Bartels, W. Gautschi and C. Witzgall Third edition, Texts in Applied Mathematics, 12, Springer, New York, 2002

Problem 16

Consider the matrix

$$A := \begin{pmatrix} 4 & 0 & -2 & 0 \\ 0 & 9 & 3 & -3 \\ -2 & 3 & 11 & -1 \\ 0 & -3 & -1 & 5 \end{pmatrix} \in \mathbb{R}^{4\times 4}.$$

(a) Show that the matrix A is positive definite.

(b) Determine a Cholesky decomposition of the matrix A.

(c) Calculate the solution of the system of equations $Ax = b$ with the right-hand side

$$b = (-10, -6, 12, 14)^\top \in \mathbb{R}^4.$$

Problem 17

Determine the natural cubic interpolating spline for the following values:

x_i	0	1	2
y_i	1	1	2

Problem 18

Consider the integral $I(f) := \int_1^2 \frac{1}{x} \mathrm{d}x$.

(a) Use the trapezoidal rule for $h = \frac{1}{3}$ to calculate an approximation for $I(f)$.

(b) Let $h = \frac{1}{4}$. Calculate an approximation of $I(f)$ using the Simpson's rule.

(c) How large must the number of subintervals N be chosen such that the error with the trapezoidal rule is not greater than 0,001?

Problem 19

For the parabola

$$f(t) = (t - \alpha) \cdot (t - \beta)$$

the real parameters α und β are to be determined from the following table:

t_k	-1	0	1	2
f_k	-4	-6	-6	-4

(a) Formulate this task as a linear minimization problem

$$\|Ax - b\|_2^2 \to \min$$

with an appropriate matrix A and vector b.

(b) Solve part (a) using the normal equation and determine α and β.

Problem 20

The system of equations

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{1}{2} \end{pmatrix}$$

is to be solved using both the Jacobi method and the Gauss-Seidel method.

(a) For both methods, provide the respective iteration matrix and its spectral radius. Justify that both methods converge independently of the choice of the initial vector.

(b) Starting with the initial value $x_0 = (0, 1)^\top$, perform two iterations for each method.

(c) Calculate the relative error of the second iterated value in each method compared to the exact solution of the system.

1.4 Stochastics

Recommended literature:

- A. Klenke, *Probability theory—a comprehensive course*, third edition, Universitext, Springer, 2020

- A. N. Shiryaev, *Probability. 1*, Graduate Texts in Mathematics, 95, Springer, New York, 2016
- H. Bauer, *Probability theory*, De Gruyter Studies in Mathematics, 23, de Gruyter, Berlin, 1996

Problem 21

Let X be a real-valued random variable on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ that is distributed according to the exponential distribution (with arbitrary parameter $\lambda \in (0,\infty)$). Show that $\mathbb{P}[\{X > t+s\}|\{X > t\}] = \mathbb{P}[\{X > s\}]$ for all $t, s \in (0,\infty)$.

Problem 22

Let X_1 and η be two independent real-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Assume that X_1 is distributed according to the standard normal distribution and η is distributed according to the Bernoulli distribution with parameter $1/2$. Let $X_2 := (2\eta - 1)X_1$. Show:

(i) X_2 is a standard normally distributed random variable.

(ii) X_1 and X_2 are uncorrelated.

(iii) X_1 and X_2 are not independent.

(iv) The joint distribution $\mathbb{P}_{(X_1,X_2)}$ of X_1 and X_2 is not a (bivariate) normal distribution.

Problem 23

Let X_n, $n \in \mathbb{N}$, be independent real-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with $\mathbb{P}_{X_n} = \mathrm{B}_{1,p_n}$ (Bernoulli distribution) for some $p_n \in (0,1)$, $n \in \mathbb{N}$. For each of the following statements, find an equivalent condition on the sequence $(p_n)_{n\in\mathbb{N}}$:

(i) X_n converges in probability to 0 as $n \to \infty$.

(ii) X_n converges in $\mathcal{L}^p$ to 0 as $n \to \infty$.

(iii) X_n converges $\mathbb{P}$-a.s. to 0 as $n \to \infty$.

Problem 24

For any $m \in \mathbb{R}$ use δ_m to denote the Dirac measure at m. For any $m \in \mathbb{R}$ and $v \in (0,\infty)$ use $\mathrm{N}_{m,v}$ to denote the univariate normal distribution with mean m and variance v. Show that the following statement holds for any $m \in \mathbb{R}$ and any sequence $(v_n)_{n\in\mathbb{N}}$ in $(0,\infty)$ with $\lim_{n\to\infty} v_n = 0$:

(i) N_{m,v_n} converges weakly to δ_m as $n \to \infty$.

Let $(X_j)_{j\in\mathbb{N}}$ be a sequence of i.i.d. real-valued random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with expectation $\mathbb{E}[X_1] = 0$ and variance $\mathbb{V}\mathrm{ar}[X_1] = 1$. Set $S_n := \sum_{j=1}^n X_j$, $n \in \mathbb{N}$. Show:

(ii) $\sqrt{n}\frac{S_n}{\sum_{j=1}^n X_j^2}$ converges in distribution to $\mathrm{N}_{0,1}$.

Here the denominator is set to 1 when $\sum_{j=1}^n X_j^2 = 0$.

2 Advanced courses

Knowledge of material in some advanced courses is an asset.

2.1 Algebra

Recommended literature:

- Siegfried Bosch, *Algebra - From the Viewpoint of Galois Theory*, Birkhäuser, 2018
- Serge Lang, *Algebra*, Springer, 2002

Problem 25

Let G, H be groups and $\varphi : G \to H$ a group homomorphism.

a) Which of the following statements is true? (Give a proof or a counterexample.)
   i) The image $\varphi(N)$ of a normal subgroup $N \subseteq G$ is normal.
   ii) The preimage $\varphi^{-1}(N)$ of a normal subgroup $N \subseteq H$ is normal.
b) Show that each subgroup U of G of index 2 is a normal subgroup.
   Hint *Let $g \in G$. What happens if $gU = U$ and what happens if $gU \neq U$?*

Problem 26

Let $\varphi : R \to S$ be a ring homomorphism. Show the following statements:

a) If $q \subseteq S$ is a prime ideal, then $\varphi^{-1}(q)$ is also a prime ideal.
b) Give an example of φ and a maximal ideal m in S, such that $\varphi^{-1}(m)$ is not maximal.
c) If a and b are ideals in R and p is a prime ideal in R with $ab \subseteq p$ then $a \subseteq p$ or $b \subseteq p$.

Problem 27

Let $\mathbb{F}_3$ be the field with 3 elements and let $f = X^2 + 1 \in \mathbb{F}_3[X]$.

a) Show that $K = \mathbb{F}_3[X]/(f)$ is a field. Determine a basis of K as $\mathbb{F}_3$-vector-space. How many elements has K?
b) What is the inverse of $X + 1$ in K?
c) Write the polynomial $X^8 - 1 \in \mathbb{Z}[X]$ as product of irreducible polynomials.
d) Show that K has a primitive eight root of unity, i.e. there is a $u \in K$ with $u^8 = 1$ and 8 is minimal with this property.

2.2 Complex analysis

Recommended literature:

- J. B. Conway, *Functions of one complex variable*, second edition, Graduate Texts in Mathematics, 11, Springer, New York-Berlin, 1978

Problem 28
Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant entire function. Show that $f(\mathbb{C})$ is dense in $\mathbb{C}$.

Problem 29
Let $\emptyset \neq D \subset \mathbb{C}$ be open and let $\mathcal{O}(D)$ denote the ring of holomorphic functions on D. Show that $\mathcal{O}(D)$ is an integral domain if and only if D is connected.

Problem 30
Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic with $f(0) = 0$. Show that the series

$$\sum_{n=0}^{\infty} f(z^n)$$

converges uniformly on compact subsets of $\mathbb{D}$.

Problem 31
Let m and n be integers satisfying $0 \leq m \leq n - 2$. Show that

$$\int_0^{\infty} \frac{x^m}{1 + x^n}\, dx = \frac{\pi}{n \sin(\frac{m+1}{n}\pi)},$$

for instance using the method of residues.

Problem 32
Let $f, g : \mathbb{C} \to \mathbb{C}$ be entire functions satisfying $f^2 + g^2 = 1$. Show that there exists an entire function $h : \mathbb{C} \to \mathbb{C}$ such that $f = \cos h$ and $g = \sin h$.

2.3 Measure theory

Recommended literature:

- D. L. Cohn, *Measure theory*, second edition, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser/Springer, New York, 2013

Problem 33
Let $(X, \mathcal{A}, \mu)$ be a measure space and let $g : X \to [0, \infty]$ be a measurable function. Define

$$\nu : \mathcal{A} \to [0, \infty], \quad \nu(A) = \int_A g\, d\mu.$$

Show:

(a) ν is measure.

(b) A measurable function $f : X \to \mathbb{R}$ is ν-integrable if and only if $\int_X |f| g \, d\mu < \infty$. In this case,

$$\int_X f \, d\nu = \int_X fg \, d\mu.$$

Problem 34

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable. Show that the derivative f' is Borel measurable.

Problem 35

Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $f : X \to [0, \infty)$ be measurable.

(a) Show that

$$\{(x, t) : X \times \mathbb{R} : 0 \leq t < f(x)\}$$

belongs to the product σ-algebra of $\mathcal{A}$ with the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

(b) Show that

$$\int_X f d\mu = \int_{[0,\infty)} \mu(\{x \in X : f(x) > t\}) \, dt.$$